Monday, January 25, 2021

Recall the Gauss sum for a Dirichlet character $\chi \pmod q$:

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a)\, e\left(\frac{a}{q}\right).$$

Theorem 9.5: If $\chi \pmod q$ is a character, and $(n,q)=1$, then

$$(*) \quad \chi(n)\,\tau(\bar{\chi}) = \sum_{a=1}^{q} \bar{\chi}(a)\, e\left(\frac{an}{q}\right)$$

In particular, when $n=-1$,

$$\chi(-1)\,\tau(\bar{\chi}) = \sum_{a=1}^{q} \bar{\chi}(a)\, e\left(\frac{-a}{q}\right)$$

$$= \sum_{a=1}^{q} \overline{\chi(a)}\; \overline{e\left(\frac{a}{q}\right)} = \overline{\tau(\chi)}.$$

We saw in Theorem 9.7 that $(*)$ holds even when $(n,q)>1$ if χ is primitive.

Recall also that $\log(1-z)^{-1} = \sum_{k=1}^{\infty} \frac{z^k}{k}$ for $|z|<1$. Claim: this series also converges when $|z|=1$, $z \neq 1$.

Dirichlet's test: If b_n is decreasing to 0, and $\sum_{n \le x} a_n$ is uniformly bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Here, $a_k = z^k$ and $b_n = \frac{1}{k}$;

$$\sum_{n \le N} z^k = \frac{z - z^{N+1}}{1-z} \quad ; \text{ when } |z|=1,\ z \neq 1.$$

$$\ll \frac{1}{|1-z|}.$$

Let's examine $L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$ for $\chi \pmod q$ primitive, $q>1$. By Theorem 9.5,

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\tau(\bar\chi)} \sum_{a=1}^{q} \bar\chi(a) e(an/q)$$

$$= \frac{1}{\tau(\bar\chi)} \sum_{a=1}^{q} \bar\chi(a) \sum_{n=1}^{\infty} \frac{1}{n} e(an/q)$$

$$= \frac{1}{\tau(\bar\chi)} \sum_{a=1}^{q-1} \bar\chi(a) \log(1-e(an/q))^{-1}.$$

Using $1 - e(\theta) = -2ie(\theta/2)\left(\frac{e(\frac{\theta}{2})-e(-\frac{\theta}{2})}{2i}\right)$

$$= -2i\, e\left(\frac{\theta}{2}\right) \sin\left(2\pi\frac{\theta}{2}\right),$$

we set

~~$L(1,\chi) = \frac{1}{\tau(\bar\chi)} \sum_{a=1}^{q-1} \bar\chi(a)\left(-2i\, e\left(\frac{an}{2q}\right)\sin\left(\pi\frac{an}{q}\right)\right)$,~~

$$L(1,\chi) = \frac{-1}{\tau(\bar\chi)} \sum_{a=1}^{q-1} \bar\chi(a)\left(\log(-2i) + \frac{a\cancel{n}}{\cancel{2}q}\pi i + \log\sin\left(\pi\frac{a\cancel{n}}{q}\right)\right)$$

$$= -\frac{1}{\tau(\bar\chi)}\left(\sum_{a=1}^{q-1} \bar\chi(a) \log\sin\frac{\pi a}{q} + i\frac{\pi}{q}\sum_{a=1}^{q-1}\bar\chi(a)\, a\right)$$

If we replace a by $q-a$ in both sums, we also get

$$L(1,\chi) = -\frac{1}{\tau(\bar\chi)}\left(\chi(-1)\sum_{a=1}^{q-1}\bar\chi(a)\log\sin\frac{\pi a}{q} + i\frac{\pi}{q}\sum_{a=1}^{q-1}(-\chi(-1))\bar\chi(a)\, a\right).$$

- If $\chi(-1)=1$, then the imaginary part must be 0.
- If $\chi(-1)=-1$, then the real part must be 0.
- $q = |\tau(\chi)|^2 = \tau(\chi)\overline{\tau(\chi)} = \chi(-1)\tau(\chi)\tau(\bar\chi)$.

$$L(1,\chi) = -\frac{1}{\tau(\bar{\chi})}\left(\chi(-1)\sum_{a=1}^{q-1}\bar{\chi}(a)\log\sin\frac{\pi a}{q}\right.$$
$$\left. + i\frac{\pi}{q}\sum_{a=1}^{q-1}(-\chi(-1))\bar{\chi}(a)a\right).$$

- If $\chi(-1)=1$, then the imaginary part must be 0.
- If $\chi(-1)=-1$, then the real part must be 0.
- $q = |\tau(\chi)|^2 = \tau(\chi)\overline{\tau(\chi)} = \chi(-1)\tau(\chi)\tau(\bar{\chi})$.

Theorem 9.9: Let $\chi \pmod q$ be primitive, $q>1$. If $\chi(-1)=1$,

$$L(1,\chi) = -\frac{\tau(\chi)}{q}\sum_{a=1}^{q-1}\bar{\chi}(a)\log\sin\frac{\pi a}{q},$$

while if $\chi(-1)=-1$,

$$L(1,\chi) = \frac{i\pi\tau(\chi)}{q^2}\sum_{a=1}^{q-1}a\bar{\chi}(a).$$

Examples: • $\chi \pmod 4$, $\chi(1)=1$, $\chi(3)=-1$.

$$\tau(\chi) = \sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right)$$
$$= \chi(1)e\left(\tfrac{1}{4}\right) + \chi(3)e\left(\tfrac{3}{4}\right)$$
$$= i + (-1)(-i) = 2i.$$

$$L(1,\chi) = \frac{i\pi}{4^2}\cdot 2i\,(1\cdot 1 + 3(-1))$$
$$= \frac{\pi}{4}.$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

• $\chi \pmod 3$, $\chi(1)=1$, $\underset{\chi(2)}{\chi(-1)} = -1$.

$$\tau(\chi) = \chi(1)e\left(\tfrac{1}{3}\right) + \chi(2)e\left(\tfrac{2}{3}\right)$$
$$= \frac{-1+\sqrt{3}i}{2} - \left(\frac{-1-\sqrt{3}i}{2}\right)$$
$$= i\sqrt{3}.$$

If $\chi(-1)=1$,

$$L(1,\chi) = -\frac{\tau(\chi)}{q}\sum_{a=1}^{q-1}\bar{\chi}(a)\log\sin\frac{\pi a}{q},$$

while if $\chi(-1)=-1$,

$$L(1,\chi) = \frac{i\pi\,\tau(\chi)}{q^2}\sum_{a=1}^{q-1} a\,\bar{\chi}(a).$$

($\chi \pmod 3$):

$$L(1,\chi) = \frac{i\pi(i\sqrt{3})}{3^2}\left(1+2(-1)\right)$$

$$= \frac{\pi}{3\sqrt{3}}.$$

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\cdots$$

- $\chi \pmod 5$, $\chi(1)=1$, $\chi(2)=\chi(3)=-1$, $\chi(4)=1$.

$$L(1,\chi) = \frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}}{2}.$$

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\cdots$$

- $\chi \pmod 7$, $\chi(1,2,4)=1$, $\chi(3,5,6)=-1$.

$$L(1,\chi) = \frac{i\pi\,\tau(\chi)}{7^2}\left(1+2-3+4-5-6\right)$$

$$= \cdots = \frac{\pi}{\sqrt{7}}.$$

Theorem 4.9: If χ is a nonprincipal character, then $L(1,\chi)\neq 0$.

Assuming Theorem 4.9 for the moment:

If $(a,q)=1$: $\displaystyle\sum_{\chi \pmod q}\bar{\chi}(a)\chi(n) = \begin{cases}\phi(q), & \text{if } n\equiv a \pmod q,\\ 0, & \text{if } n\not\equiv a \pmod q.\end{cases}$

So

$$\sum_{\substack{n\geq 1\\ n\equiv a \pmod q}}\frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\left(\frac{1}{\phi(q)}\sum_{\chi \pmod q}\bar{\chi}(a)\chi(n)\right)$$

(for $\sigma>1$)

$$= \frac{1}{\phi(q)}\sum_{\chi \pmod q}\bar{\chi}(a)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\chi(n)$$

$$= \frac{-1}{\phi(q)}\sum_{\chi \pmod q}\bar{\chi}(a)\,\frac{L'}{L}(s,\chi).$$

$$\sum_{n\equiv a \pmod q} \frac{\Lambda(n)}{n^s} = \frac{-1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \frac{L'}{L}(s,\chi).$$

$$= -\frac{1}{\phi(q)}\left(\frac{L'}{L}(s,\chi_0) + \sum_{\chi\neq\chi_0} \overline{\chi}(a)\frac{L'}{L}(s,\chi)\right).$$

Let $s \in \mathbb{R}$, $s \to 1^+$. Since $L(s,\chi_0)$ has a simple pole at $s=1$, we have

$$-\frac{1}{\phi(q)}\frac{L'}{L}(s,\chi_0) = \frac{1}{\phi(q)(s-1)} + O_q(1).$$

Since each other $L(s,\chi)$ is analytic and nonzero at $s=1$, we have

$$\frac{L'}{L}(s,\chi) = O_q(1) \text{ for } \chi\neq\chi_0,$$
s near 1.

So as $s \to 1^+$

$$\sum_{n\equiv a \pmod q} \frac{\Lambda(n)}{n^s} = \frac{1}{\phi(q)(s-1)} + O_q(1)$$

$\to \infty$ as $s \to 1^+$.

In other words,

$$\sum_{n\equiv a \pmod q} \frac{\Lambda(n)}{n} = +\infty.$$

For the proper prime powers:

$$\sum_{k=2}^{\infty} \sum_{p^k\equiv a \pmod q} \frac{\log p}{p^k} \leq \sum_p \log p \sum_{k=2}^{\infty} \frac{1}{p^k}$$

$$= \sum_p \frac{\log p}{p(p-1)} < \infty.$$

Corollary 4.10 (Dirichlet)

If $(a,q)=1$, then

$$\sum_{p\equiv a \pmod q} \frac{\log p}{p} \text{ diverges}$$

(In particular, ∞ many primes $p\equiv a \pmod q$)

If $(a,q)=1$, then $\sum_{p\equiv a \pmod q} \frac{\log p}{p}$ diverges.

You can read Theorem 4.11 and Corollary 4.12 -- some examples:

- $$\sum_{\substack{p\le x \\ p\equiv a \pmod q}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_q(1)$$

- $$\sum_{\substack{p\le x \\ p\equiv a \pmod q}} \frac{1}{p} = \frac{1}{\phi(q)} \log\log x + b(q,a) + O_q\left(\frac{1}{\log x}\right).$$

Proof of Theorem 4.9 ($L(1,\chi)\neq 0$)

It turns out that real (quadratic) characters are more difficult to understand where their zeros are.

Proof for complex χ: consider

$$\prod_{\chi \pmod q} L(s,\chi) = \exp\left(\sum_{\chi} \log L(s,\chi)\right)$$

$$= \exp\left(\sum_{\chi \pmod q} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} \chi(n) n^{-s}\right)$$

($\sigma>1$)

$$= \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \sum_{\chi \pmod q} \chi(n)\right)$$

$$= \exp\left(\underbrace{\sum_{n\equiv 1 \pmod q} \phi(q) \frac{\Lambda(n)}{\log n} n^{-s}}_{\text{nonnegative}}\right).$$

If $s=\sigma>1$,

Hence $\prod_{\chi \pmod q} L(s,\chi) \ge 1$ when $\sigma>1$.

It's impossible for two $L(s,\chi)$'s to have zeros at $s=1$, since then $\prod_{\chi} L(s,\chi)$ would have a zero there.

(contradiction as $s\to 1^+$).

If χ is complex, then $\overline{\chi} \neq \chi$

where $L(1,\chi) = 0 \Leftrightarrow L(1,\overline{\chi}) = \overline{L(1,\chi)}$
$= 0$.

So no $L(s,\chi)$ with χ complex can have a zero at $s=1$.

Wednesday: we'll prove $L(1,\chi) \neq 0$ when χ is quadratic