$$\sum_{n\equiv a \pmod q} \frac{\Lambda(n)}{n^s} = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \bar{\chi}(a) \sum_{n=1}^{\infty} \chi(n)\frac{\Lambda(n)}{n^s}$$

Wednesday, January 27

Landau's theorem (Theorem 1.7, MV)

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ have abscissa of converge σ_c. If $a_n \geq 0$ for all n, then $\alpha(s)$ has a singularity at $s = \sigma_c$.

Equivalently: if $\alpha(s)$ has an analytic continuation to $\{\mathrm{Re}(s) > \sigma_0\}$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ converges on $\{\mathrm{Re}\, s > \sigma_0\}$.

• Proof that $L(1,\chi) \neq 0$ when χ is quadratic

Lemma: Let $r = \chi * 1$, that is,

$$r(n) = \sum_{d \mid n} \chi(d)\, 1\!\left(\tfrac{n}{d}\right) = \sum_{d \mid n} \chi(d).$$

Then $r(n) \geq 0$, and $r(n^2) \geq 1$ for all n.

Proof: Since $\chi, 1$ are both multiplicative, so is r; so it suffices to check on prime powers.

	1	p	p^2	p^3	p^4	p^5	…
1	1	1	1	1	1	1	
if $\chi(p)=1$:	1	1	1	1	1	1	
$r=\chi*1$	1	2	3	4	5	6	
if $\chi(p)=-1$	1	-1	1	-1	1	-1	
$r=\chi*1$	1	0	1	0	1	0	

Proof that $L(1,\chi)\neq 0$ for χ quadratic

Let $f(s) = \zeta(s)L(s,\chi)$, so that

$$\underset{r=\chi*1}{f(s)} = \underset{1}{\zeta(s)}\,\underset{\chi}{L(s,\chi)}$$

$$f(s)=\sum_{n=1}^{\infty} r(n)n^{-s}.$$

If $L(1,\chi)=0$, then $f(s)$ is analytic for $\sigma>0$. By Landau's theorem, $\sum_{n=1}^{\infty} r(n)n^{-s}$ converge for $\sigma>0$.

On the other hand

$$f(\tfrac12) = \sum_{n=1}^{\infty} r(n)n^{-\frac12}$$

$$\geq \sum_{m=1}^{\infty} r(m^2)(m^2)^{-1/2} \geq \sum_{m=1}^{\infty} 1\cdot m^{-1}=\infty.$$

Poisson summation formula (Appendix D.2)

Let $f\in L^1(\mathbb{R})$, and define its Fourier transform

$$\hat f(t) = \int_{\mathbb{R}} f(x)e(-tx)\,dx.$$

If f is of bounded variation on $\mathbb{R}$, then if f is continuous we have

$$\sum_{n\in\mathbb{Z}} f(n) = \lim_{M\to\infty}\sum_{m=-M}^{M}\hat f(m).$$

One can show that if

$$f(x) = e^{-\pi x^2 z},$$

then

$$\hat f(x) = z^{-1/2}e^{-\pi x^2/z}.$$

Theorem 10.1: For $\alpha \in \mathbb{R}$ and z with $\operatorname{Re} z > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 z} = z^{-1/2} \sum_{k=-\infty}^{\infty} e(k\alpha) e^{-\pi k^2/z}$$

and

$$\sum_{n=-\infty}^{\infty} (n+\alpha) e^{-\pi(n+\alpha)^2 z} = -i z^{-3/2} \sum_{k=-\infty}^{\infty} k e(k\alpha) e^{-\pi k^2/z}.$$

(The branch $z^{1/2}$ is determined by $1^{1/2}=1$.)

Definition: For $\operatorname{Re} z > 0$ and $\chi \pmod q$, let

$$\theta_0(z,\chi) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 z/q}$$

$$\theta_1(z,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n) e^{-\pi n^2 z/q}.$$

Remark: If $\chi(-1)=1$, then $\theta_1(z,\chi)$ is identically 0; if $\chi(-1)=-1$, then $\theta_0(z,\chi)$ is identically 0.

Also, $$|\theta_0(z,\chi)| \le \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-\pi n^2 \operatorname{Re} z/q}$$

$$\le 2\sum_{n=1}^{\infty} e^{-\pi n \operatorname{Re} z/q}$$

$$\ll_{q,\delta} e^{-\pi \operatorname{Re} z/q}$$

uniformly for $\operatorname{Re} z > \delta$.

and $\chi \pmod q$

Definition: For $\operatorname{Re} z > 0$, let

$$\theta_0(z,\chi) = \sum_{n=-\infty}^{\infty} \chi(n)\, e^{-\pi n^2 z/q}$$

$$\theta_1(z,\chi) = \sum_{n=-\infty}^{\infty} n\,\chi(n)\, e^{-\pi n^2 z/q}.$$

Theorem 10.8: For $\operatorname{Re} z > 0$ and $\chi \pmod q$ primitive,

$$\theta_0(z,\chi) = \frac{\tau(\chi)}{q^{1/2}}\, z^{-1/2}\, \theta_0\!\left(\tfrac{1}{z}, \bar\chi\right)$$

$$\theta_1(z,\chi) = \frac{\tau(\chi)}{i q^{1/2}}\, z^{-3/2}\, \theta_1\!\left(\tfrac{1}{z}, \bar\chi\right).$$

Proof: Since χ has period q,

$$\theta_0(z,\chi) = \sum_{a \,(\mathrm{mod}\, q)} \chi(a) \sum_{\substack{n \equiv a \,(\mathrm{mod}\, q)\\ n \in \mathbb{Z}}} e^{-\pi n^2 z/q}$$

$$= \sum_a \chi(a) \sum_{m \in \mathbb{Z}} e^{-\pi (mq+a)^2 z/q}$$

which Theorem 10.1 applies to with $\alpha = \frac{a}{q}$. Same factor

$$\sum_{a \,(\mathrm{mod}\, q)} \chi(a) \cdot e(ka) = \sum_{a \,(\mathrm{mod}\, q)} \chi(a)\, e\!\left(k\tfrac{a}{q}\right)$$

arises

$$\| \quad \bar\chi(k)\, \tau(\chi).$$

Similarly for θ_1. //

Notation: $\kappa = \kappa(\chi) = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$

So $\chi(-1) = (-1)^\kappa$.

Also define $\varepsilon(\chi) = \dfrac{\tau(\chi)}{i^\kappa \sqrt{q}}$.

Thus for any primitive $\chi \pmod q$,

$$\theta_\kappa(z,\chi) = \frac{\varepsilon(\chi)}{z^{1/2+\kappa}}\, \theta_\kappa\!\left(\tfrac{1}{z}, \bar\chi\right).$$

Note: when χ is primitive

$|\varepsilon(\chi)| = 1$, and $\varepsilon(\chi)\varepsilon(\bar{\chi}) = 1$.

$\Uparrow$

comes from $\overline{\tau(\chi)} = \chi(-1)\tau(\bar{\chi})$.

and $|\tau(\chi)| = \sqrt{q}$.

Goal for start of Monday:

prove this functional equation

for $L(s,\chi)$, when χ is primitive and $q > 1$:

Define $\xi(s,\chi) = L(s,\chi)\Gamma\left(\frac{s+k}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{s+k}{2}}$.

Then $\xi(s,\chi)$ is entire, and

$$\xi(s,\chi) = \varepsilon(\chi)\,\xi(1-s,\bar{\chi}).$$