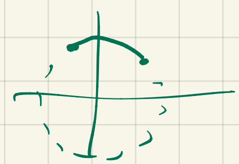
Monday, March 1

- Kyle's 1st presentation on Mertens's Conjecture

Consider $e(y) = e^{2\pi i y}$, a function from $\mathbb{R} \to \mathbb{C}$.
If we fix an interval on the unit circle, say $I = (e^{2\pi i \alpha}, e^{2\pi i \beta})$



then in every interval of length 1 in $\mathbb{R}$, the measure of those y such that $e(y) \in I$ equals $\beta - \alpha$. Hence

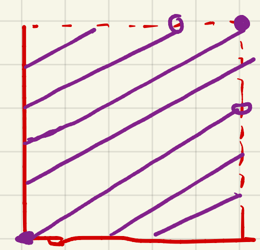
$$\lim_{Y \to \infty} \frac{1}{Y} \operatorname{meas}\{0 \le y \le Y : e(y) \in I\}$$
$$= \lim_{Y \to \infty} \frac{1}{Y}\left((\beta-\alpha)Y + O(1)\right) = \beta - \alpha.$$

Thus $e(y)$ is uniformly distributed on S^1; it has a limiting distribution, a measure ν, on S^1 — namely the uniform (Haar) measure on S^1.

Note: $x^{2\pi i} = e(\log x)$ is not uniformly distributed on S^1. (exercise)
But it is "logarithmically uniformly distributed" (set $y = \log x$).
(Same cov: $\frac{M(x)}{\sqrt{x}} \to \frac{M(e^u)}{e^{u/2}}$ $(u = \log x)$)

Now fix $\gamma_1, \gamma_2 \in \mathbb{R}$ and consider the ray $\{(t\gamma_1, t\gamma_2) : t \in \mathbb{R}\}$ in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Ex: $\gamma_1 = 5$, $\gamma_2 = 3$. This ray is a one-dimensional subtorus of $\mathbb{T}^2$.



$F(t) = \frac{1}{2}e(5t) + \frac{1}{3}e(3t)$

$t \in \mathbb{R} \longmapsto (5t, 3t) \in \mathbb{T}^2 \longmapsto \mathbb{C}.$

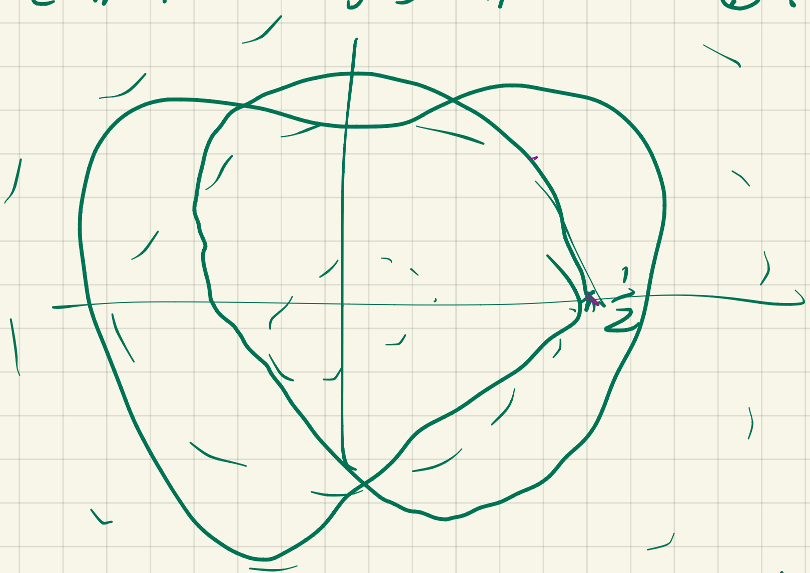
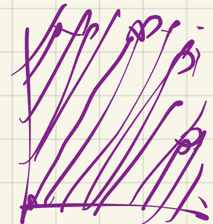


Image of $F(t)$ will have a limiting distribution function supported on this curve.

Note: $(r_1, r_2) = (5, 3)$ or $(5\sqrt{2}, 3\sqrt{2})$ or $\left(\frac{5}{e}, \frac{3}{e}\right)$ all give the same limiting distribution.

Now consider $(r_1, r_2) = (1, \sqrt{2})$



Turns out: the ray $(t, t\sqrt{2})$ is dense in $\mathbb{T}^2$ (Kronecker's theorem).

"inhomogeneous Diophantine approximation"

Even stronger: Kronecker–Weyl theorem: the limiting distribution of the ray is Haar measure on $\mathbb{T}^2$.

Kronecker–Weyl (full): Let $r_1, \dots, r_n \in \mathbb{R}$.

Let R = {all $\mathbb{Q}$-linear relations among $r_1, \dots, r_n$}.

Let A be the subtorus of $\mathbb{T}^n$ defined by R

$$\begin{tikzcd} V \cap \mathbb{Z}^n \arrow[r, hook] \arrow[d] & V \arrow[r, two heads] \arrow[d, hook] & V/(V \cap \mathbb{Z}^n) \cong \mathbb{T}^k \arrow[d, hook] \\ \mathbb{Z}^n \arrow[r, hook] & \mathbb{R}^n \arrow[r, two heads] & \mathbb{T}^n \end{tikzcd}$$

$V = \{x \in \mathbb{R}^n : x \text{ satisfies all relations in } R\}$

Then the ray $\{t(r_1, \dots, r_n) : t \in \mathbb{R}\}$ is uniformly distributed on A.

- In particular, if $\{r_1, \dots, r_n\}$ is linearly independent over $\mathbb{Q}$, then the ray is uniformly distributed on $\mathbb{T}^n$.

Equivalent characterizations of equidistribution:

- The ray $\{t(r_1, \dots, r_n) : t \in \mathbb{R}\}$ is equidist'd in a subtorus A;
- If $f : \mathbb{T}^n \to \mathbb{C}$ is continuous, then

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y f(tr_1, \dots, tr_n)\, dt = \int_A f(x)\, d\,\mathrm{Haar}(x).$$

- Define $\Psi : \mathbb{T}^n \to \mathbb{R}^r$ by

$$\Psi(t_1, \dots, t_n) = 2\,\mathrm{Re} \sum_{j=1}^{n} z_j e(t_j).$$

(fixed $z_j \in \mathbb{R}^j$)

Then the function $\Psi(tr_1, \dots, tr_n) : \mathbb{R} \to \mathbb{R}^r$ has a limiting distribution: if f is a bounded continuous function from $\mathbb{R}^r \to \mathbb{R}$, then

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y f(\Psi(tr_1, \dots, tr_n))\, dt = \int_{\mathbb{R}^r} f(x)\, d\nu\, x$$

for some measure ν.

(ν ultimately depends on the subtorus A, hence on the set R of $\mathbb{Q}$-linear relations among the r_j)

(also depends on the z_j)

$\varphi(t_1, t_2) = \frac{1}{2} e(t_1) \; \frac{1}{3} e(t_2)$

Ratner's equidistribution theorem