Wednesday, March 17

Recall these definitions:

- $$\delta_{q;a,b} = \lim_{x\to\infty} \frac{1}{\log x} \int_{[2,x]\cap R} \frac{dt}{t},$$

where $R = \{z \geq 2 : \pi(z;q,a) > \pi(z;q,b)\}$

- $\rho(q) = \#\{x \pmod q : x^2 \equiv 1 \pmod q\}$,
- $$b(\chi) = \sum_{\substack{\gamma\in\mathbb{R} \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{1}{\frac{1}{4}+\gamma^2}$$
- $$V(q;a,b) = \sum_{\chi \pmod q}{}' |\chi(b)-\chi(a)|^2 b(\chi)$$

We showed: assume GRH and LI. If a is a nonsquare $\pmod q$ and b is a square $\pmod q$,

$$\delta_{q;a,b} = \frac{1}{2} + \Pr\left(0 < N(0,1) < \frac{\rho(q)}{\sqrt{V(q;a,b)}}\right) + O\left(\frac{1}{V(q;a,b)}\right).$$

- If a is a square and b is a nonsquare, same formula for $\delta_{q;a,b}$ except the $+$ turns into $-$
- If a,b are both squares or both nonsquares, then $\delta_{q;a,b} = \frac{1}{2}$.

Remarks: If $r \in (\mathbb{Z}/q\mathbb{Z})^\times$, then

$$|\chi(br) - \chi(ar)| = |\chi(b)\chi(r) - \chi(a)\chi(r)| = |\chi(b) - \chi(a)|;$$

therefore $V(q;a,b) = V(q;ar,br)$.

So, when racing a nonsquare vs. a square, we can assume that $b \equiv 1 \pmod q$ (just take $r \equiv b^{-1} \pmod q$).

- Similarly, $|\chi(1) - \chi(a^{-1})| = |\chi(1) - \overline{\chi(a)}| = |\chi(1) - \chi(a)|$.

Hence $\delta_{q;a,1} = \delta_{q;a^{-1},1}$.

Let's estimate $\Pr(0 < N(0,1) < \delta)$ for δ small.

$\Pr = F(\delta)$ where $F(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$.

By first-year calculus,

$$F(\delta) = F(0) + F'(0)\delta + \frac{F''(c)}{2!}\delta^2 \text{ for some } c \in (0,\delta).$$

Also $F'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and

$$F''(x) = \frac{-1}{\sqrt{2\pi}} x \, e^{-x^2/2} \ll |x|.$$

So $F(\delta) = \frac{1}{\sqrt{2\pi}}\delta + O(\delta^3)$.

With $\delta = \frac{\rho(q)}{\sqrt{V(q;a,b)}}$, we get

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{V(q;a,b)} + \cancel{\frac{\rho(q)^3}{V(q;a,b)^{3/2}}}\right).$$

- $$b(\chi) = \sum_{\substack{\gamma \in \mathbb{R} \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{1}{\frac{1}{4}+\gamma^2}$$

- $$V(q;a,b) = \sum_{\chi \pmod q} |\chi(b)-\chi(a)|^2 \, b(\chi)$$

We know: $b(\chi) = \log q^* + O(\log\log q)$, where q^* is the conductor of χ (χ is induced by primitive $\chi^* \pmod{q^*}$).

Thus

$$V(q;a,b) = \sum_{\chi \pmod q} |\chi(b)-\chi(a)|^2 \left(\log q^* + O(\log\log q)\right)$$

$$= \sum_{\chi \pmod q} |\chi(b)-\chi(a)|^2 \log q^* + O(\phi(q)\log\log q).$$

I'll refer to "Inequities ..." with Fiorilli.

Lemma 3.2: For $q \in \mathbb{N}$, and for $s \mid q$, $1 \le s < q$,

✓. $$\sum_{d \mid q} \Lambda\left(\frac{q}{d}\right) \phi(d) = \phi(q) \sum_{p \mid q} \frac{\log p}{p-1}$$

✓. $$\sum_{d \mid s} \Lambda\left(\frac{q}{d}\right) \phi(d) = \phi(q) \frac{\Lambda(q/s)}{\phi(q/s)}.$$

Proposition 3.3: For $q \in \mathbb{N}$, and for $a \in (\mathbb{Z}/q\mathbb{Z})^\times \setminus \{1\}$,

✓. $$\sum_{\chi \,(\mathrm{mod}\, q)} \log q^* = \phi(q)\left(\log q - \sum_{p \mid q} \frac{\log p}{p-1}\right)$$

✓ $$\sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \log q^* = -\phi(q) \frac{\Lambda(q/(q, a-1))}{\phi(q/(q,a-1))}.$$

if $a \equiv 1 \pmod q$ then we're already done

Proof: We start by evaluating

maybe $a \equiv 1$

$$\sum_{d \mid q} \Lambda\left(\frac{q}{d}\right) \sum_{\chi \,(\mathrm{mod}\, d)} \chi(a)$$

$$= \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \sum_{\substack{d \mid q \\ q^* \mid d}} \Lambda\left(\frac{q}{d}\right)$$

$\left(c = \frac{q}{d}\right)$ $$= \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \sum_{c \mid q/q^*} \Lambda(c)$$

$$= \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \log\left(\frac{q}{q^*}\right)$$

$$= \log q \cdot \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) - \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \log q^*.$$

For the words,

$$\sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \log q^* = \log q \cdot \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) - \sum_{d \mid q} \Lambda\left(\frac{q}{d}\right) \sum_{\chi \,(\mathrm{mod}\, d)} \chi(a).$$

If $a \not\equiv 1 \pmod q$,

$$\sum_{\chi \pmod q} \chi(a) \log q^* = \cancel{\phi(q) \sum_{\chi \pmod q} \chi(a)}$$

$$- \sum_{d \mid q} \Lambda\left(\frac{q}{d}\right) \sum_{\chi \pmod d} \chi(a)$$

$$= - \sum_{d \mid q} \Lambda\left(\frac{q}{d}\right) \phi(d)\, \iota_d(a)$$

[where $\iota_m(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod m \\ 0, & \text{otherwise.} \end{cases}$]

$$= - \sum_{\substack{d \mid q \\ d \mid (a-1)}} \Lambda\left(\frac{q}{d}\right) \phi(d)$$

$$= - \phi(q) \frac{\Lambda(q/(q, a-1))}{\phi(q/(q, a-1))}$$ by Lemma 3.2.

Now consider

$$\sum_{\chi \pmod q} |\chi(b) - \chi(a)|^2 \log q^*$$

$$= \sum_{\chi \pmod q} (\chi(b) - \chi(a))(\overline{\chi(b)} - \overline{\chi(a)}) \log q^*$$

$$= \sum_{\chi \pmod q} (2 - \chi(ab^{-1}) - \chi(ba^{-1})) \log q^*$$

From Proposition 3.3,

$$= 2\phi(q)\left(\log q - \sum_{p \mid q} \frac{\log p}{p-1}\right) + \phi(q) \frac{\Lambda(q/(q, ab^{-1}-1))}{\phi(q/(q, ab^{-1}-1))} + \left(\text{same with } ab^{-1} \to ba^{-1}\right)$$

Note: since $(a,q) = (b,q) = 1$,

$(q, ab^{-1}-1) = (q, b(ab^{-1}-1)) = (q, a-b)$

and same for $(q, ba^{-1}-1)$.

Thus $\sum_{\chi \pmod q} |\chi(b) - \chi(a)|^2 \log q^* =$

$$2\phi(q)\left(\log q - \sum_{p \mid q} \frac{\log p}{p-1} + \frac{\Lambda(q/(q, a-b))}{\phi(q/(q, a-b))}\right),$$

Also, $\frac{\log t}{t-1}$ is decreasing, so

$$\sum_{p|q} \frac{\log p}{p-1} \le \sum_{i=1}^{w(q)} \frac{\log p_i}{p_i - 1}$$

$$\ll \sum_{p < w \log w} \frac{\log p}{p} \sim \log(w \log w)$$

$$\ll \log w \ll \log\log q.$$

Since $w = w(q) \le \frac{\log q}{\log 2}$.

This proves that

$$V(q;a,b) = 2\phi(q) \log q + O(\phi(q) \log\log q).$$

Therefore $= (2\phi(q)\log q)\left(1 + O\left(\frac{\log\log q}{\log q}\right)\right)$.

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(\frac{1}{V} + \frac{\rho^3}{V^{3/2}}\right)$$

$$= \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi\phi(q)\log q}}\left(1 + O\left(\frac{\log\log q}{\log q}\right)\right).$$

<u>Next</u>: look more closely at the dependence of $V(q;a,1)$ and $\delta_{q;a,1}$ on the non-sq. a.

- Exact formula for $b(\chi)$: $\boxed{\text{(Vorhauer)}}$ assuming GRH

$$b(\chi) = \sum_{\substack{\gamma \in \mathbb{R} \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{1}{\frac{1}{4}+\gamma^2}$$

$$b(\chi) = \log\left(\frac{q^*}{\pi}\right) - \gamma_0 - (1+\chi(-1))\log 2 + 2\,\mathrm{Re}\,\frac{L'}{L}(1,\chi^*).$$

$(\chi \neq \chi_0)$

<u>Proposition 3.1</u>: Let a, b be distinct reduced residue classes $(\mathrm{mod}\ q)$. Then

- $$\sum_{\chi\ (\mathrm{mod}\ q)} |\chi(b)-\chi(a)|^2 = 2\phi(q)$$

- If $c \not\equiv 1 \pmod q$, then

$$\sum_{\chi\ (\mathrm{mod}\ q)} |\chi(b)-\chi(a)|^2 \chi(c) = -\phi(q)\left(\iota_q(cab^{-1}) + \iota_q(cba^{-1})\right).$$

Theorem 1.4: Assume GRH. Let $a, b \in (\mathbb{Z}/q\mathbb{Z})^\times$ be distinct. Then

$$V(q;a,b) = 2\phi(q)\Big(\mathcal{L}(q) + K_q(a-b) + \iota_q(-ab^{-1})\log 2\Big) + \underline{2M^*(q;a,b)},$$

where

$$\mathcal{L}(q) = \log q - \sum_{p \mid q} \frac{\log p}{p-1} + \frac{\Lambda(q)}{\phi(q)} - (\gamma_0 + \log 2\pi)$$

$$K_q(n) = \frac{\Lambda(q/(q,n))}{\phi(q/(q,n))} - \frac{\Lambda(q)}{\phi(q)}.$$

$$M^*(q;a,b) = \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} |\chi(b) - \chi(a)|^2 \frac{L'}{L}(1, \chi^*)$$