Wednesday, March 24

- References to M–Fiorilli, "Inequities..."

the variance $V(q;a,b) = \sum_{\chi \,(\mathrm{mod}\, q)} |\chi(b)-\chi(a)|^2 b(\chi)$,

where $b(\chi) = \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14+\gamma^2}$

Theorem 1.4: On GRH,

$$V(q;a,b) = 2\phi(q)\Big(\mathcal{L}(q) + K_q(a-b) + \iota_q(-ab^{-1})\log 2\Big) + 2M^*(q;a,b),$$

where:

- $\mathcal{L}(q) = \log q - \sum_{p\mid q} \frac{\log p}{p-1} + \frac{\Lambda(q)}{\phi(q)} - (\gamma_0 + \log 2\pi)$
  - $\mathcal{L}(q) = \log q + O(\log\log q)$ $\boxed{\mathcal{L}(q)>0 \text{ when } q \geq 43}$
  - if q is prime then $\mathcal{L}(q) = \log\left(\frac{q}{2\pi e^{\gamma_0}}\right)$
- $\iota_q(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod q \\ 0, & \text{otherwise} \end{cases}$
- $K_q(a-b) = \frac{\Lambda(q/(q,a-b))}{\phi(q/(q,a-b))} - \frac{\Lambda(q)}{\phi(q)}$
- $M^*(q;a,b) = \sum_{\chi \,(\mathrm{mod}\, q)} |\chi(b)-\chi(a)|^2 \frac{L'}{L}(1,\chi^*)$

Theorem 1.7: Assume GRH. Let r_1 and r_2 be the least positive residues of $ab^{-1} \pmod q$ and $ba^{-1} \pmod q$. Then

$$M^*(q;a,b) = \phi(q)\left(\frac{\Lambda(r_1)}{r_1} + \frac{\Lambda(r_2)}{r_2} + H(q;a,b) + O\left(\frac{\log^2 q}{q}\right)\right),$$

where:

- if $p^\nu \| q$, set $h(q;p,r) = \frac{1}{\phi(p^\nu)} \frac{\log p}{p^{e(q;p,r)}}$,

  where $e(q;p,r) =$ smallest $e \in \mathbb{N}$ with $p^e \equiv r^{-1} \pmod{\frac{q}{p^\nu}}$. (or ∞)
- $H(q;a,b) = \sum_{p^\nu \| q} \left(h(q;p,ab^{-1}) + h(q;p,ba^{-1})\right)$

[$H(q;a,b)$ comes from changing $\frac{L'}{L}(1,\chi^*)$ to $\frac{L'}{L}(1,\chi)$.]

Recall Theorem 1.1: on GRH and LI,

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(q^{-3/2+\varepsilon}\right).$$

(when a is a nonsquare, b is a square)

Taking the linear approximation to $\frac{\rho(q)}{\sqrt{2\pi x}}$ at $x = \mathcal{L}(q)$:

Corollary 1.4: (same assumptions)

$$\delta(q;a,b) = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi\phi(q)\mathcal{L}(q)}}\left(1 - \frac{\Delta(q;a,b)}{2\mathcal{L}(q)} + O\left(\frac{1}{\log^2 q}\right)\right),$$

where

$$\Delta(q;a,b) = K_q(a-b) + \iota_q(-ab^{-1})\log 2 + \frac{\Lambda(r_1)}{r_1} + \frac{\Lambda(r_2)}{r_2} + H(q;a,b).$$

$(r_1 \equiv ab^{-1},\ r_2 \equiv ba^{-1})$

- $\Delta(q;a,b) \geq 0$ and $\Delta(q;a,b) \ll 1$
- $\Delta(q;a,b) \neq 0$ when:
  - $a \equiv -b \pmod q$
  - $a \equiv b$ (modulo a large divisor of q)
  - r_1 or r_2 is a prime power
  - (something about H)
- $\Delta(q;a,b) > 0$ means $\delta_{q;a,b}$ is smaller

Example: $q = 163$.

- the first 12 primes and -1 are all quadratic nonresidues $\pmod{163}$.

$\delta_{q;a,1}$ where a is a nonsquare $\pmod{163}$

- $\Delta(163;-1,1) \geq \log 2$
- if a is prime, $\Delta(163;a,1) \geq \frac{\log a}{a}$.

Theorem 1.10: Assume GRH and LI.

- Convention: when we write $\delta_{q;n,1}$, we implicitly restrict q to moduli where $(n,q)=1$ and n is a nonsquare $\pmod q$.

(a) If $a \neq 1$, then $\delta_{q;-1,1} < \delta_{q;a,1}$ for all but finitely many q.

(b) If a is a prime power and $a' \neq -1$ is not a prime power, then $\delta_{q;a,1} < \delta_{q;a',1}$ for all but finitely many q.

(c) If a and a' are prime powers with $\frac{\Lambda(a)}{a} > \frac{\Lambda(a')}{a'}$, then $\delta_{q;a,1} < \delta_{q;a',1}$ for all but finitely many q.

Notation: $E(x;q,a) = \dfrac{\phi(q)\pi(x;q,a) - \pi(x)}{\sqrt{x}/\log x}$

Bays–Hudson "mirror image phenomenon":

(mod 4).

Let $q \equiv 3 \pmod 4$ be prime.

Then if b is a square $\pmod q$, we observe:

When $\pi(x;q,b)$ is unusually small, $\pi(x;q,-b)$ tends to be unusually large.

Let's investigate $E(x;q,b) + E(x;q,-b)$

More generally, we look at $E(x;q,a) + E(x;q,b)$

- under GRH and LI, this has a limiting log distribution with variance

$$V^+(q;a,b) = \sum_{\chi \pmod q} |\chi(a) + \chi(b)|^2 b(\chi).$$

evaluating V^+ ...

equation (4.5)

$$V^+(q;a,b) = 2\phi(q)\Big(\log q - \sum_{p \mid q} \frac{\log p}{p-1} - (\gamma_0 + \log 2\pi)\Big)$$

$$- K_q(a-b) \boxed{- \iota_q(-ab^{-1}) \log 2)} + 2M^+(q;a,b)$$

$$-4b(\chi_0).$$

where $M^+(q;a,b) = \sum_{\chi (\mathrm{mod}\ q)} |\chi(a)+\chi(b)|^2 \frac{L'}{L}(1,\chi^*).$

- when $a \equiv -b \pmod q$, the variance is smaller.

M–N 5, "Inclusive prime number races"

- Let $\Gamma(\chi) = \{\gamma > 0 : L(\frac{1}{2}+i\gamma, \chi) = 0\}$ and $\Gamma(q) = \bigcup_{\chi (\mathrm{mod}\ q)} \Gamma(\chi)$.
- We say $\gamma \in \Gamma(q)$ is "self-sufficient" if $\gamma \notin \mathrm{span}_{\mathbb{Q}}(\Gamma(q) \setminus \{\gamma\})$.
- LI $\Leftrightarrow$ every $\gamma \in \Gamma(q)$ being self-suff.

Theorem 1.5: Assume GRH.

- Suppose $\bigcup_{\substack{\chi (\mathrm{mod}\ q) \\ \chi(a) \neq \chi(b)}} \Gamma(\chi)$ contains 3 self-sufficient zeros. Then $\delta_{q;a,b}$ exists.
- Moreover, there exists $W(q)$ such that if
$$\boxed{\sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi(a) \neq \chi(b)}} \sum_{\substack{\gamma \in \Gamma(\chi) \\ \gamma \text{ self-s.}}} \frac{1}{\gamma}} > W(q),$$
then $\delta_{q;a,b} > 0$. $= \Pr\Big(E(x;q,a) - E(x;q,b) > 0\Big)$
- If $\boxed{\text{sum}}$ diverges, then
$$\Pr\big(E(x;q,a) - E(x;q,b) \in (\alpha,\beta)\big) > 0$$
for any $(\alpha,\beta) \subseteq \mathbb{R}$.

Lucile Devin, "Chebyshev's bias for analytic L-functions":

Theorem 5.4 (LD)

Suppose there exists $\varepsilon > 0$ such that for $T \gg_\varepsilon 1$,

$$\left(\bigcup_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi(a) \neq \chi(b)}} \Gamma(\chi) \right) \cap \left(T^{\frac{1}{2}-\varepsilon}, T\right)$$

contains ≥ 1 self-sufficient zero.

Then $\delta_{q;a,b}$ exists.