Wednesday, March 31

(start: Devang's presentation on the Class Number Formula)

Toy example in probability:

Let X_1, X_2, X_3 be independent random variables, uniform on $[-50,50]$, $[-6,6]$, $[-6,6]$.

- Note that $Pr(X_i > X_j) = \frac{1}{2}$ for every $i \neq j$, $i,j \in \{1,2,3\}$.
- Three-way orderings:
  - Suppose $X_1 > 6$. (44%)
    - $X_1 > X_2 > X_3$ 22%
    - $X_1 > X_3 > X_2$ 22%
  - Suppose $X_1 < -6$ (44%)
    - $X_2 > X_3 > X_1$ 22%
    - $X_3 > X_2 > X_1$ 22%
  - ① If we condition on $|X_1| \leq 6$: (12%)
    - each of the six orderings is equally likely by symmetry.

So overall, the three-way race gives:

$X_1 > X_2 > X_3$	24%
$X_1 > X_3 > X_2$	24%
$X_2 > X_3 > X_1$	24%
$X_3 > X_2 > X_1$	24%
$X_2 > X_1 > X_3$	2%
$X_3 > X_1 > X_2$	2%.

Next example: X_1, X_2, X_3 independent normal random variables with mean 0, variances $\alpha^2, \beta^2, \gamma^2$.

- By symmetry: $Pr(X_i > X_j > X_k) = Pr(X_k > X_j > X_i)$.
- Turns out: one can calculate $Pr(X_1 > X_2 > X_3)$ exactly.

Next example: X_1, X_2, X_3 independent normal random variables with mean 0, variances $\alpha^2, \beta^2, \gamma^2$.

- Density function of X_1 is $\frac{1}{\alpha\sqrt{2\pi}} e^{-x^2/2\alpha^2}$ (etc.)

We want to calculate $\Pr(X_1<X_2<X_3) =$

$$\int_{-\infty}^{\infty}\int_{x}^{\infty}\int_{y}^{\infty} \frac{1}{\alpha\beta\gamma(2\pi)^{3/2}}\, e^{-\frac{x^2}{2\alpha^2}-\frac{y^2}{2\beta^2}-\frac{z^2}{2\gamma^2}}\,dz\,dy\,dx.$$

Change of variables $y=x+s$, $z=x+s+t$:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\alpha\beta\gamma(2\pi)^{3/2}}\, e^{-\frac{x^2}{2\alpha^2}-\frac{(x+s)^2}{2\beta^2}-\frac{(x+s+t)^2}{2\gamma^2}}\,dt\,ds\,dx.$$

Fact: $$\int_{-\infty}^{\infty} e^{-ax^2-bx}\,dx = \int_{-\infty}^{\infty} e^{-a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}}\,dx$$
$$= e^{b^2/4a}\int_{-\infty}^{\infty} e^{-a\overset{x}{\cancel{\left(x-\frac{b}{2a}\right)}}^2}\,dx = e^{b^2/4a}\sqrt{\frac{\pi}{a}}.$$

Thus

$$\frac{1}{\alpha\beta\gamma(2\pi)^{3/2}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}-\frac{(x+s)^2}{2\beta^2}-\frac{(x+s+t)^2}{2\gamma^2}}\,dx =$$

$$\frac{1}{2\pi\tau}\exp\left(-\frac{s^2(\beta^2+\gamma^2)-2st(\alpha^2+\gamma^2)+t^2(\alpha^2+\beta^2)}{2\tau^2}\right),$$

where $\tau^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$.

- Remaining integral is $\int_0^\infty\int_0^\infty \quad dt\,ds$, which is an "orthant probability": $\Pr[$some 2-d normal variable lies in $[0,\infty)^2]$.

- possible to diagonalize the quadratic form to $e^{-(v^2+w^2)}$ by linear change of variables; region of integration turns into a sector in $\mathbb{R}^2$ — trivial to integrate in polar coordinates.

Final answer:

$$\Pr(X_1<X_2<X_3) = \frac{1}{2\pi}\arctan\frac{\tau}{\beta^2}.$$

$$\Pr(X_1<X_2<X_3) = \frac{1}{2\pi}\arctan\frac{\sqrt{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}}{\beta^2} \quad (*)$$

If $\alpha^2, \beta^2, \gamma^2$ are all close to each other:

write $\alpha^2=\sigma^2(1+\varepsilon)$, $\beta^2=\sigma^2(1+\delta)$, $\gamma^2=\sigma^2(1+\eta)$. Linear approximation to $(*)$ at $(\varepsilon,\delta,\eta)=(0,0,0)$ gives

$$\Pr(X_1<X_2<X_3) = \frac{1}{6}+\frac{\varepsilon-2\delta+\eta}{8\pi\sqrt{3}}+O(\varepsilon^2+\delta^2+\eta^2).$$

- Literature contains closed forms for 3- and 4-dimensional orthant probabilities of multivariate normal random variables — nondiagonal ok ($X_1,\dots,X_4$ don't have to be independent).

- When we estimate $\delta_{q;a_1,\dots,a_r}$ by $\Pr(X_1>\dots>X_r)$, we can set roughly

$$\delta_{q;a_1,a_2,a_3,a_4} = (\text{orthant probability in closed form}) + O\left(\frac{1}{q}\right).$$

Example: Lamzouri, "Prime number races with three or more competitors":

Corollary 2.3:

$$\delta_{q;a_1,a_2,a_3} = \frac{1}{6}+\frac{1}{4\sqrt{\pi}}\frac{c(q,a_3)-c(q,a_1)}{\sqrt{V(q)}} + \frac{1}{4\pi\sqrt{3}}\frac{B_q(a_1,a_2)-2B_q(a_1,a_3)+B_q(a_2,a_3)}{V(q)} + O(\mathrm{blah}^2).$$

$c(q,a) = -1+\#\{b^2\equiv a \pmod q\}$

$$V(q) = 2\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14+\gamma^2} \sim 2\phi(q)\log q.$$

$$B_q(a,b) = 2\sum_{\chi}\left(\chi(ab^{-1})+\chi(ba^{-1})\right)\sum_{\gamma>0}\frac{1}{\frac14+\gamma^2}.$$