Monday, April 3

Recall that given some function $\Delta_0(x)$, we recursively define $\Delta_k(x) = \int_0^x \frac{\Delta_{k-1}(t)}{t}\,dt$.

Lemma: The number of sign changes of $\Delta_{k-1}(x)$ on $[0,A]$ is at least as many as the number of sign changes of $\Delta_k(x)$ on $[0,A]$.

Proof: If we have $0 = x_0 < x_1 < x_2 < \cdots < x_w$ where $\mathrm{sign}(\Delta_k(x_j)) = (-1)^j$. Then

$$0 < (-1)^j\left(\Delta_k(x_j) - \Delta_k(x_{j-1})\right)$$

$$= (-1)^j \int_{x_{j-1}}^{x_j} \Delta_{k-1}(t)\,\frac{dt}{t};$$

Thus $(-1)^j \Delta_{k-1}(t)$ must be positive for some $y_j \in [x_{j-1}, x_j]$. //

We apply this (for example) to

$$\Delta_0(x) = \Delta^{\psi}(x) = \psi(x) - x = \sum_{n \le x} (\Lambda(n) - 1)$$

$$= -\sum_{\rho} \frac{x^\rho}{\rho} + \text{error}.$$

Then

$$\Delta_k(x) = \sum_{n \le x} (\Lambda(n) - 1)\left(\log \frac{x}{n}\right)^k$$

$$= -\sum_{\rho} \frac{x^\rho}{\rho^{k+1}} + \text{error}_k.$$

In particular, let $\rho_1 = \frac{1}{2} + \overbrace{14.135\ldots}^{\gamma_1} i$; then

$$\left|\Delta_k(x) + 2\,\mathrm{Re}\,\frac{x^{\rho_1}}{\rho_1^{k+1}}\right| \le \left|2\,\mathrm{Re}\sum_{\substack{\rho \ne \rho_1 \\ \mathrm{Im}\,\rho > 0}} \frac{x^\rho}{\rho^{k+1}} + \text{error}_k\right|$$

If RH is true then

$$\left| \frac{\Delta_k(x)}{\sqrt{x}} + 2\,\mathrm{Re}\, \frac{x^{i\gamma_1}}{\rho_1^{k+1}} \right| \le 2\,\mathrm{Re} \sum_{\gamma > \gamma_1}' \frac{x^{i\gamma}}{\rho^{k+1}} + \frac{\text{error}_k}{\sqrt{x}}$$

$$\le 2\,\mathrm{Re} \sum_{\gamma > \gamma_1}' \frac{1}{\rho^{k+1}} + \frac{\text{error}_k}{\sqrt{x}}.$$

It turns out that when $k=3$,

$$2\,\mathrm{Re}\, \frac{1}{\rho_1^4} \approx \frac{1}{20{,}000} \quad \text{while}$$

$$2\,\mathrm{Re} \sum_{\gamma > \gamma_1}' \frac{1}{\rho^4} \approx \frac{1}{41{,}000}.$$

So if we choose $\{x_j\}$ such that

$\mathrm{Re}\, x_j^{i\gamma_1} = \cos(\gamma_1 \log x_j) = \pm 1$, the

we get sign changes of $2\,\mathrm{Re}\, \frac{x^{i\gamma_1}}{\rho^4}$

and hence sign changes of

$\frac{\Delta_k(x)}{\sqrt{x}} \Rightarrow$ sign changes $\Delta_0(x) = \Delta^{\psi}(x)$.

We get $\#\{$sign changes of $\psi(x) - x$ up to $X\}$

$$\ge \frac{\log X}{\pi/\gamma_1} + O(1).$$

(assuming RH)

Results from the literature:

- Let $\Theta = \sup\{\mathrm{Re}\, \rho\}$, and set

$$\gamma_0 = \inf\{\gamma > 0 : \zeta(\Theta + i\gamma) = 0\}.$$

Note: maybe $\gamma_0 = \infty$.

Define $W^{\psi}(X)$ to be the # of sign changes of $\psi(x) - x$ up to X.

Then

$$\liminf_{X \to \infty} \frac{W^{\psi}(X)}{\log X} \ge \frac{\gamma_0}{\pi}.$$

- Knapowski, 1985:

$\#\{\text{sign changes of } \theta(x) - x \text{ up to } X\}$ $\gg \log X$ (constant is ineffective)

and the same for $\pi(x) - li(x)$

- Schlage-Puchta, 2004:

similar result for sign changes of

$$\pi(x;q,1) - \max_{\substack{(a,q)=1 \\ a \neq 1}} \pi(x;q,a)$$

(and min)

(# sign changes is $\gg \dfrac{\log X}{f(q)}$) on GRH.

How many sign changes do we expect?

Think about $\pi(x;4,3) - \pi(x;4,1)$.

Model by a symmetric random walk

$$S_k = \sum_{j=1}^{k} X_j \quad \text{where } X_j = \pm 1$$

(50/50 prob).

Feller: # sign changes of $\{S_k\}$

- sign change at S_j means $S_{j-1} S_{j+1} = -1$.

$$\Pr\left(\#\{\text{sign changes up to } 2n\} = r\right) = \frac{1}{2^{2n-2}} \binom{2n-1}{n-1-r}.$$

As $n \to \infty$, this is modeled by a normal random variable with mean 0, variance proportional to n.

We can derive:

- expected number of sign changes up to N is
$$\sim \sqrt{\frac{2}{\pi}}\sqrt{N} \approx 0.399\sqrt{N}$$
- median is $\approx 0.337\sqrt{N}$.

Maybe we conjecture
$$\#\{\text{sign changes of } \pi(x;4,3)-\pi(x;4,1) \text{ up to } X\} \overset{?}{\asymp} \sqrt{\frac{X}{\log X}} \; ?$$

Exhaustive 2-way prime number races (infinitely many sign changes): unconditional results.

- Littlewood, Stark, Snead: all two-way races with $q \le 100$
- Kátai: if the $L(s,\chi)$ (mod q) have no real nontrivial zeros (Haselgrove's condition, +C), then any race square vs square, nonsquare vs nonsquare is exhaustive.
- Knapowski/Turán: assume +C, all 2-way races involving $\pi(x;q,1)$ are exhaustive.
- Vorhauer: all 2-way races involving (if +C) $\pi(x;q,-1)$ are exhaustive.