Wednesday, February 1

Warm-up inequality: for any quadratic character $\chi \pmod q$, and $\sigma > 1$,

$$-\frac{L'}{L}(\sigma,\chi_0) - \frac{L'}{L}(\sigma,\chi) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \Lambda(n) n^{-\sigma}(1+\chi(n)) \geq 0.$$

Lemma 11.2 (MV): For any character $\chi \pmod q$, any $\sigma > 1$, and any $t \in \mathbb{R}$:

$$\operatorname{Re}\Big(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\Big) \geq 0. \quad (*)$$

Proof: The left-hand side equals

$$\operatorname{Re}\left(\sum_{\substack{n\geq 1 \\ (n,q)=1}} \frac{\Lambda(n)}{n^\sigma}\left(3 + 4\frac{\chi(n)}{n^{it}} + \frac{\chi^2(n)}{n^{2it}}\right)\right).$$

Write $\chi(n)/n^{it} = e^{i\theta_n}$ for some $\theta_n \in \mathbb{R}$: this equals

$$\sum_{\substack{n\geq 1 \\ (n,q)=1}} \frac{\Lambda(n)}{n^\sigma} \operatorname{Re}(3 + 4e^{i\theta_n} + e^{2i\theta_n})$$

$$= \sum_{(n,q)=1} \frac{\Lambda(n)}{n^\sigma}(3 + 4\cos\theta_n + \cos 2\theta_n)$$

$$= \sum_{(n,q)=1} \frac{\Lambda(n)}{n^\sigma} 2(1+\cos\theta_n)^2 \geq 0. \quad //$$

Consequence: $L(s,\chi) \neq 0$ when $\sigma = 1$.

- As $\sigma \to 1^+$, $L(\sigma,\chi_0)$ has a simple pole, so $-\frac{L'}{L}(\sigma,\chi_0)$ has a simple pole of residue 1. Thus $-3\frac{L'}{L}(\sigma,\chi_0) \sim \frac{3}{\sigma-1}$ as $\sigma \to 1^+$.
- If $L(1+it,\chi) = 0$, then $-\frac{L'}{L}(\sigma+it,\chi)$ would have a simple pole of residue -1; thus $-4\frac{L'}{L}(\sigma+it,\chi) \sim \frac{-4}{\sigma-1}$ as $\sigma \to 1^+$.
- $-\frac{L'}{L}(\sigma+2it,\chi^2)$ is bounded above since there's no pole (if either $t \neq 0$ or $\chi^2 \neq \chi_0$).

Hence LHS of $(*)$ would be $\leq -\frac{1}{\sigma-1} + O(1)$, contradiction.

Some tools from complex analysis, about analytic functions f on closed disks.

- "Jensen's lemma" (Lemma 6.1 in MV): upper bound for the # of zeros of f in a smaller disc, in terms of $|f(0)|$ and $\max |f(z)|$ (and also the disks' radii).
- "Borel–Carathéodory lemma" (Lemma 6.2): upper bounds for $|f|$ and $|f'|$ in a smaller disk when $f(0)=0$, in terms of $\max (\mathrm{Re}\, f(z))$ (and the radii).

Lemma 6.3 (MV): Suppose $f(z)$ is analytic on $|z|\le 1$, that $|f(z)|\le M$ there, and $f(0)\ne 0$. Fix $0<r<R<1$. Then for $|z|\le r$,

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{K} \frac{1}{z-z_k} + O_{R,r}\left(\log \frac{M}{|f(0)|}\right)$$

where $z_1, \dots, z_K$ are the zeros of f on the medium disk $|z|\le R$.

Notes:
- If f has multiple zeros, then we list them multiple times.
- If f is a polynomial, then $\frac{f'}{f}(z) = \sum_{k=1}^{K} \frac{1}{z-z_k}$.

Consequences:

Lemma 6.4: If $|t|\ge \frac{7}{8}$ and $\frac{5}{6}\le \sigma \le 2$, then

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O(\cancel{\log t}\ \log \tau),$$

where the sum is over zeros, $\rho=\beta+i\gamma$, of $\zeta(s)$ for which $|\rho-(\frac{3}{2}+it)|\le \frac{5}{6}$.

- not really $\log t$; we need something like $\max\{\log|t|, 1\}$ or $\log(|t|+4)$

MV define $\tau = |t|+4$.

This result for $\zeta(s)$ implies an analogous result for $L(s,\chi_0)$ since

$$L(s,\chi_0) = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right).$$

Lemma 11.1 (MV): When $\frac{5}{6} \leq \sigma \leq 2$,

$$-\frac{L'}{L}(s,\chi_0) = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log(q\tau)) \quad (**)$$

where ρ runs over zeros of $L(s,\chi_0)$ satisfying $\left|\rho - \left(\frac{3}{2}+it\right)\right| \leq \frac{5}{6}$.

It also follows from Lemma 6.3:

Lemma 11.1: if $\chi \neq \chi_0$, we get the same expression $(**)$ without the $\frac{1}{s-1}$.

Observation: if $s = \sigma + it$ with $\sigma > 1$, and $\rho = \beta + i\gamma$ has $\beta < 1$, then $\mathrm{Re}(s-\rho) = \sigma - \beta > 0$ and so $\mathrm{Re}\left(\frac{1}{s-\rho}\right) > 0$.

Hence for upper bounds on $\mathrm{Re}\left(-\frac{L'}{L}(s,\chi)\right)$, we can throw some or all of these ρ away:

if $\chi \neq \chi_0$,

$$\mathrm{Re}\left(-\frac{L'}{L}(s,\chi)\right) \leq O(\log q\tau).$$

Or, if ρ is a special zero near s, then

$$\mathrm{Re}\left(-\frac{L'}{L}(s,\chi)\right) \leq -\mathrm{Re}\,\frac{1}{s-\rho} + O(\log q\tau)$$

Theorem 11.3 — the "zero-free region" for Dirichlet L-functions:

There exists an absolute constant $c>0$ such that, for any Dirichlet character χ (mod q), the region

$$\left\{ s=\sigma+it : \sigma > 1-\frac{c}{\log q\tau} \right\}$$

contains no zeros of $L(s,\chi)$, unless χ is quadratic, in which case there might be one zero $\rho_1=\beta_1$ on the real axis (so $\beta_1<1$).

hypothetical "exceptional zero"

Sketch of the proof:

Case 1: χ is complex. Let $\rho_0=\beta_0+i\gamma_0$ be a zero of $L(s,\chi)$.
Look at $s=1+\delta+i\gamma_0$ for some $\delta>0$.

- $\mathrm{Re}\left(-\frac{L'}{L}(1+\delta,\chi_0)\right) \leq \frac{1}{\delta}+O(\log q)$
- $\mathrm{Re}\left(-\frac{L'}{L}(1+\delta+i\gamma_0,\chi)\right) \leq -\frac{1}{1+\delta-\beta_0}+O(\log q\tau)$
- $\mathrm{Re}\left(-\frac{L'}{L}(1+\delta+2i\gamma_0,\chi^2)\right) \leq O(\log q\tau)$

Hence

$$0 \leq \mathrm{Re}\left(-3\frac{L'}{L}(1+\delta,\chi_0)-4\frac{L'}{L}(1+\delta+i\gamma_0,\chi)-\frac{L'}{L}(1+\delta+2i\gamma_0,\chi^2)\right)$$
$$\leq \frac{3}{\delta}-\frac{4}{1+\delta-\beta_0}+O(\log q\tau).$$

- incompatible with nonnegativity as $\delta\to 1^+$, if $1-\beta_0$ is small.

We get a bound of the form

$$1-\beta_0 \geq \frac{c}{\log q\gamma_0}.$$