Friday, February 10

Recall: $\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n)$

- Throughout today, $(a,q)=1$.
- We may assume $q \le x$.

Truncated explicit formula

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi}(a) \sum_{|\gamma| \le T} \frac{x^\rho}{\rho}$$
$$+ \frac{1}{\phi(q)} \sum_{\chi \pmod q} C(\chi) + O\left(\log x + \frac{x \log^2 x}{T}\right),$$

where $\sum_\rho$ is over nontrivial zeros of $L(s,\chi)$.

Zero-free region: In these sums, we have $\beta \le 1 - \frac{c}{\log qT}$ for some absolute constant c. $\Rightarrow$

$$|x^\rho| \le x \cdot x^{-c/\log qT} \quad \ldots$$

... except maybe an exceptional zero, β_1, to some quadratic character χ_1.

- If β_1 exists, then two "error" terms could be large:
  - $L(1-\beta_1,\chi) < 0$ by functional eq. and $1-\beta_1$ close to 0. So $\frac{x^{1-\beta_1}}{1-\beta_1}$ is large. Also, $C(\chi) = \frac{L'}{L}(1,\chi) + \text{stuff}$ and $\frac{L'}{L}(1,\chi_1) = \frac{1}{1-\beta_1} + O(\log q)$.

Luckily, these two terms basically cancel each other out.

Another estimate: $\sum_{|\gamma| \le T} \frac{1}{|\rho|} \ll \log^2 T \ll \log^2 x$

since we will choose $T \le x$.

Inserting these estimates gives:

$$\psi(x;q,a) = \frac{x}{\phi(q)}\left(-\frac{x^{\beta_1}}{1-\beta_1} \text{ if } \beta_1 \text{ exists}\right)$$
$$+ O\left(x\log^2 x\left(\frac{1}{T} + x^{-c/\log qT}\right)\right).$$

Choose T so that $\frac{1}{T} = x^{-c/\log qT}$

$$\Rightarrow T \asymp \exp(c\sqrt{\log x}).$$

We conclude:

Corollary 11.17 (MV) — Page

If there is no exceptional zero then
$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x\exp(-c\sqrt{\log x})).$$

If β_1 is an exceptional zero of $L(s,\chi_1)$ then
$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\chi_1(a)}{\phi(q)}\frac{x^{\beta_1}}{\beta_1}$$
$$+ O(x\exp(-c\sqrt{\log x})).$$

Exploration of the consequences of β_1:

[Recall: $1-\beta_1 \gg (q^{\frac{1}{2}}\log^2 q)^{-1}$ effectively

$1-\beta_1 \gg_\varepsilon q^{-\varepsilon}$ (Siegel) ineffectively]

Thought experiment: Imagine q is prime and $\chi_1(a) = \left(\frac{a}{q}\right)$.

Suppose $\beta_1 = 1 - \frac{1}{\sqrt{q}}$ is exceptional.

- Case 1: $\chi_1(a) = -1$ (a is a quadratic nonresidue (mod q)).

Then $\frac{x}{\phi(q)} - \frac{\chi_1(a)}{\phi(q)}\frac{x^{\beta_1}}{\beta_1} = \frac{x}{\phi(q)} + \frac{x^{\beta_1}}{\beta_1\phi(q)}$

$$\asymp \frac{2x}{\phi(q)}.$$

- Case 2: $\chi_1(a) = 1$

(a is a quadratic residue (mod q)).

Then we have $\frac{1}{\phi(q)}\left(x - \frac{x^{\beta_1}}{\beta_1}\right)$. How big is this? Let $f(t) = \frac{x^t}{t}$, then the Mean Value Theorem says

$$x - \frac{x^{\beta_1}}{\beta_1} = f(1) - f(\beta_1) = f'(\xi)(1-\beta_1)$$

for some $\xi \in (\beta_1, 1)$. And

$$f'(t) = \frac{x^t}{t^2}(t\log x - 1) < \frac{x^t \log x}{t} \ll x\log x.$$

So $x - \frac{x^{\beta_1}}{\beta_1} \ll x\log x\,(1-\beta_1) = \frac{x\log x}{\sqrt{q}}$.

Hence $\psi(x;q,a) \;\hat{=}\; \frac{x}{\phi(q)}\frac{\log x}{\sqrt{q}}$, which is o(what we expect) when $\log x = o(\sqrt{q})$. So if $q > (\log x)^{2+\varepsilon}$ or $x < e^{q^{\frac{1}{2}-\varepsilon}}$: very few quadratic-residue primes.

Corollary 11.19 (Siegel–Walfisz Theorem)

For any $A > 0$, if $(a,q) = 1$ and $q \le (\log x)^A$, then

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O_A\left(x\exp(-c\sqrt{\log x})\right),$$

where the O_A constant is ineffective (if $A \ge 2$).

Note: if we assume no exceptional zeros, then we can improve to

$$q \ll \exp(c\sqrt{\log x}).$$

If we assume GRH (generalized Riemann hypothesis, $L(s,\chi) = 0 \Rightarrow \sigma \le \frac{1}{2}$) we get

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(\sqrt{x}\log^2 x)$$

$$\theta(x;q,a) = \frac{x}{\phi(q)} + O(\sqrt{x}\log^2 x)$$

$$\pi(x;q,a) = \frac{\mathrm{li}(x)}{\phi(q)} + O(\sqrt{x}\log x).$$

Consider

$$\sum_{q\le Q} \max_{y\le x} \max_{(a,q)=1} \left|\psi(y;q,a) - \frac{y}{\phi(q)}\right|. \quad (*)$$

If GRH is true then this is

$$\ll \sum_{q\le Q} \max_{y\le x} \left(y^{\frac{1}{2}} \log^2 y\right) = \sum_{q\le Q} x^{\frac{1}{2}} \log^2 x$$

$$\ll x^{\frac{1}{2}} Q \log^2 x.$$

We do know — Bombieri-Vinogradov Theorem that $(*) \ll_A x^{\frac{1}{2}} Q (\log x)^2$ uniformly for $x^{\frac{1}{2}}(\log x)^{-A} \le Q \le x^{\frac{1}{2}}$.

"GRH on average for q up to $x^{\frac{1}{2}}(\log x)^{-A}$".

Elliott-Halberstam conjecture:

$(*) \ll_{A,\varepsilon} x(\log x)^{-A}$ uniformly for $Q < x^{1-\varepsilon}$,

One implication*: there are infinitely primes p such that one of $p+2$, $p+4$, $p+6$ is also prime.

(Zhang; Maynard-Tao; Polymath)

Linnik's theorem: There is an absolute constant L such that for all $(a,q)=1$, there exists a prime $p \equiv a \pmod q$ such that $p < q^L$.

Pan (1958): $L = 5{,}448$

Heath-Brown (1992): $L = 5.5$ (effective)

Xylouris (2011): $L = 5$

GRH $\Rightarrow$ $L = 2+\varepsilon$

conjecture: $L = 1+\varepsilon$.