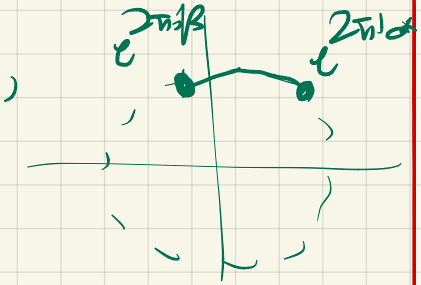
Monday, February 13

Consider $e: \mathbb{R}_{\geq 0} \to \mathbb{C}$ given by $e(y) = e^{2\pi i y}$. If we fix an arc on the unit circle, say $A = [e^{2\pi i \alpha}, e^{2\pi i \beta}]$,



then in every interval of length 1 in $\mathbb{R}$, the measure of the set of y for which $e(y) \in A$ is $\beta - \alpha$.

Hence $\lim_{Y \to \infty} \frac{1}{Y} \operatorname{meas}\{0 \leq y \leq Y : e(y) \in A\}$

$= \lim_{Y \to \infty} \frac{1}{Y}\left((\beta-\alpha)Y + O(1)\right) = \beta - \alpha.$

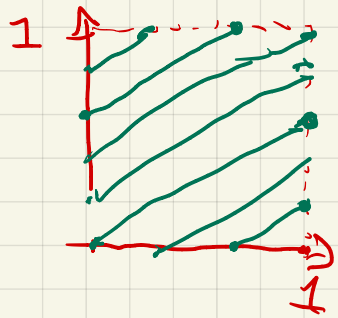
Thus $e(y)$ is uniformly distributed on $S^1 \in \mathbb{C}$ in the limit; it has a <u>limiting distribution</u> $\longrightarrow$ measure on the codomain $\mathbb{C}$. (In this case, this measure is supported on S^1 and is uniform on S^1 — Haar measure.)

<u>Note:</u> $x^{ir} = e^{ir\log x}$ is not uniformly distributed on S^1 (its limiting dist'n doesn't exist: it oscillates too slowly). But if we set $x = e^y$, so $x^{ir} = e^{iry}$, then the limiting dist'n exists.

We say that x^{ir} has a <u>limiting logarithmic distribution</u>.

Now let's fix $r_1, r_2 \in \mathbb{R}$ and consider the ray $\{(tr_1, tr_2) : t \in \mathbb{R}_{\geq 0}\}$ in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

- Example 1: $r_1 = 5$, $r_2 = 3$.

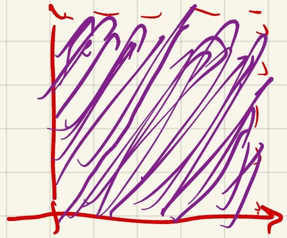


The image is constrained to a 1-dim'l subtorus of $\mathbb{T}^2$. $(3x - 5y = 0)$

The ray has limiting dist'n which is uniform on this subtorus.

<u>Note</u>: $(r_1, r_2) = (5, 3)$ or $(5\pi, 3\pi)$ or $\left(\frac{5}{e}, \frac{3}{e}\right)$ all give the same ray.

- Example 2: $r_1 = 1$, $r_2 = \sqrt{2}$.



Turns out: the ray $\{(t, t\sqrt{2})\}$ is dense in $\mathbb{T}^2$ (Kronecker theorem on inhomogeneous Diophantine approx'n.)

Even stronger: Kronecker-Weyl Thm says the limiting dist'n is uniform (Haar) on $\mathbb{T}^2$.

Kronecker–Weyl Theorem (full)

Let $r_1, \dots, r_n \in \mathbb{R}$. Let $R =$ $\{$all $\mathbb{Q}$-linear relations among $r_1, \dots, r_n\}$.

Let A be the subtorus of $\mathbb{T}^n$ defined by R. (Here V is the subspace of $\mathbb{R}^n$ defined by R.)

$$
\begin{tikzcd}
V \cap \mathbb{Z}^n \arrow[r, hook] \arrow[d, hook] & V \arrow[r, two heads] \arrow[d, hook] & V/(V \cap \mathbb{Z}^n) = A \cong \mathbb{T}^k \arrow[d, hook] \\
\mathbb{Z}^n \arrow[r, hook] & \mathbb{R}^n \arrow[r, two heads] & \mathbb{T}^n
\end{tikzcd}
$$

The the ray $\{t(r_1, \dots, r_n) : t \in \mathbb{R}_{\geq 0}\}$ is uniformly distributed on A.

In particular, if $\{r_1, \dots, r_n\}$ is linearly independent over $\mathbb{Q}$, then $R = \emptyset$ and the ray is uniformly distributed in all of $\mathbb{T}^n$.

Here "$\{t(r_1, \dots, r_n) : t \in \mathbb{R}_{\geq 0}\}$ is uniformly distributed on A" means: if $f : \mathbb{T}^n \to \mathbb{C}$ is continuous, then

$$\lim_{y \to \infty} \frac{1}{y} \int_0^y f(t(r_1, \dots, r_n))\, dt = \int_A f(a)\, d\mathrm{Haar}(a).$$

Examples of limiting dist'ns (Desmos):

$$f_1(t) = \frac{1}{5} e(5t) + \frac{1}{3} e(3t)$$

$$f_2(t) = \frac{1}{5} e(t) + \frac{1}{3} e(\sqrt{2}\, t).$$

Typical application:

Define $\psi : \mathbb{T}^n \to \mathbb{R}^r$ by

$$\psi(t_1, t_2, \dots, t_n) = 2\,\mathrm{Re} \sum_{j=1}^{n} \vec{z}_j\, e(t_j)$$

(ψ depends on fixing $\vec{z}_1, \dots, \vec{z}_n$).

Then the function $\psi(t\gamma_1, \dots, t\gamma_n) : \mathbb{R}_{\geq 0} \to \mathbb{R}^r$ has a limiting distribution:

if f is a bounded continuous function from $\mathbb{R}^r \to \mathbb{R}$, then

$$\lim_{y \to \infty} \frac{1}{y} \int_0^y f(\psi(t\gamma_1, \dots, t\gamma_n))\, dt = \int_{\mathbb{R}^r} f(x)\, d\nu(x)$$

for some measure ν (which depends on the subtorus in $\mathbb{T}^n$ defined by $\gamma_1, \dots, \gamma_n$, as well as on the $\vec{z}_j$).

This is exactly the type of function arising in explicit formulas:

$$\sum_{\chi \,(\mathrm{mod}\, q)} \vec{z}_\chi \sum_{|\gamma| \leq T} \frac{x^{i\gamma}}{\rho}$$