Feb 15 | Limiting Distributions

$$\sum_{\chi} \bar{\chi}(a) \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{x^{\rho-\frac{1}{2}}}{\rho}$$

under GRH $\rho = \frac{1}{2}+i\gamma$ $\gamma \in \mathbb{R}$

$$x^{\rho-\frac{1}{2}} = x^{i\gamma} = e^{i\gamma \log x}$$

$y = \log x$

$$\sum_{\gamma} c_\gamma e^{i\gamma y}$$

$\gamma \in \mathbb{R}$ $\forall T>0$ $\{|\gamma| \leq T\}$ is finite

$$G(y) = 2\mathrm{Re}\Big(\sum_{\gamma>0} c_\gamma e^{i\gamma y}\Big) \qquad c_{-\gamma} = \overline{c_\gamma} \in \mathbb{C}^d$$

$$G_T(y) = 2\mathrm{Re}\Big(\sum_{0<\gamma\leq T} c_\gamma e^{i\gamma y}\Big)$$

$y \mapsto e^{i\gamma y}$ periodic function

almost periodic function

Def: [Limiting distribution]

Let $F : \mathbb{R}^+ \longrightarrow E$ (measurable space)

If it exist, the limiting distribution of F is the probability measure μ on E given by $\forall B \subset E$ Borelian $\mu(B) = \lim_{Y\to\infty} \frac{1}{Y} \operatorname{meas}(y \in [0,Y] \,;\, F(y) \in B)$

$\iff$ $\forall f : E \to \mathbb{R}$ continuous bounded Lipschitz $\int_E f \, d\mu = \lim_{Y\to\infty} \frac{1}{Y} \int_0^Y f(F(y))\, dy$

$G_T : y \longmapsto \mathrm{Re}\Big(\sum_{\substack{0<\gamma\leq T \\ \text{finite}}} c_\gamma e^{i\gamma y}\Big)$ admits a limiting distribution.

if G_T was periodic of period P

$$\lim_{Y\to\infty} \frac{1}{Y} \operatorname{meas}\{y \in [0,Y],\ G_T(y) \in B\}$$

$$\lim_{Y\to\infty} \frac{1}{Y}\Big(\Big[\frac{Y}{P}\Big] \operatorname{meas}\{y \in [0,P],\ G_T(y) \in B\} + O(P)\Big) = \frac{1}{P} \operatorname{meas}\{y \in [0,P],\ G_T(y) \in B\}$$

Kronecker-Weyl theorem

Let $\gamma_1, \dots, \gamma_n \in \mathbb{R}$

$\subset \mathbb{Z}^n$

R relations between γ's

$A \in \mathbb{T}^n$ generated by R $A = \{(\theta_1, \dots, \theta_n) \in \mathbb{T}^n : \forall (k_1, \dots, k_n) \in R \quad k_1\theta_1 + \dots + k_n\theta_n = 0 \text{ in } \mathbb{T}\}$

$= R^\perp$

then the function

$\mathbb{R}^+ \longrightarrow \mathbb{T}^n$

$t \longmapsto (t\gamma_1, \dots, t\gamma_n)$

admits a limiting distribution which is the Haar measure on A.

(normalized)

$\forall h$ continuous function on $\mathbb{T}^n$ $\lim_{Y\to\infty} \frac{1}{Y} \int_0^Y h(t\gamma_1, \dots, t\gamma_n)\, dt = \int_A h \, d\mathrm{Haar}_A$

$\forall f$ continuous on $\mathbb{R}^d$

$$\lim_{Y\to\infty} \frac{1}{Y}\int_0^Y f(G_T(t))\,dt = \lim_{Y\to\infty}\frac{1}{Y}\int_0^Y \underbrace{f\left(2\mathrm{Re}\left(\sum_{n=1}^{N} c_\gamma e^{i\gamma y}\right)\right)}_{\tilde f(y\gamma_1,\dots,y\gamma_N)} dy = \int_A \tilde f \, d\mathrm{Haar}_A$$

$$\int_A \tilde f\, d\mathrm{Haar}_A = \int_{\mathbb{R}^d} f\, d\mu_T$$

μ_T a push forward of Haar_A $\square$

$\mathbb{T}^n$ duality $\mathbb{Z}^n$

$$\langle (\theta_1,\dots,\theta_n), (k_1,\dots,k_n)\rangle = e^{i2\pi(\theta_1 k_1+\dots+\theta_n k_n)}$$

Th: If $G_T \xrightarrow[T\to\infty]{L^2} G$: $\limsup_{Y\to\infty} \frac{1}{Y}\int_0^Y |G(y)-G_T(y)|^2 dy \xrightarrow[T\to\infty]{} 0$

B^2-almost periodic function

and $\forall T$ G_T admits a limiting distribution μ_T

then G admits a limiting distribution μ and $\mu_T \to \mu$

$\forall f$ continuous bounded Lipschitz $\int f d\mu_T \to \int f d\mu$

proof: f bounded continuous Lipschitz

$$\underbrace{\frac{1}{Y}\int_0^Y f(G(y))\,dy}_{\substack{\liminf \\ \limsup \\ Y\to\infty}} = \frac{1}{Y}\int_0^Y f(G_T(y))\,dy + \underbrace{\frac{1}{Y}\int_0^Y f(G(y)) - f(G_T(y))\,dy}_{\ll_f \frac{1}{Y}\int_0^Y |G(y)-G_T(y)|\,dy}$$

$\downarrow$ $\int f d\mu_T$

$$\leq \frac{1}{\sqrt{Y}}\left(\int_0^Y |G(y)-G_T(y)|^2 dy\right)^{1/2}$$

$$\leq \limsup_{Y\to\infty} \sqrt{}$$

$\square$

* Symmetry of μ or μ_T : $\mathrm{Proba}(G>0) = \mathrm{Proba}(G<0)$

$\Leftrightarrow$ $\forall f$ continuous $\int_{\mathbb{R}^d} f(-t)\,d\mu_T(t) = \int_{\mathbb{R}^d} f(t)\,d\mu_T(t)$

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y f(\underbrace{-G_T(t)}_{2\mathrm{Re}(\sum_\gamma c_\gamma e^{i\gamma t+\pi})})\,dt = \int_A \tilde f(a)\,d\mathrm{Haar}_A(a)$$

$$\int_{A+(\frac12,\dots,\frac12)} \tilde f(a)\,d\mathrm{Haar}_A =$$

$A' = A+(\frac12,\dots,\frac12)$

$\Leftrightarrow (\frac12,\dots,\frac12)\in A$.

* Smoothness (does not necessary go to the limit)

Paley-Wiener theorems: smoothness of a function / measure
$\Longleftrightarrow$ speed of decay of its Fourier transform (*) at infinity.

ex: "ties have density 0"

$y \mapsto G(y) \longrightarrow \mu$ limiting distribution

I want to show that G has probability 0 to be 0.

i.e $\mu(\{0\}) = 0$

If $\mu(\{0\}) \neq 0 \rightarrow$ mass at 0 $\rightsquigarrow$ Dirac Delta

Fourier transform of Dirac Delta

$$\hat{\delta}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi t} d\delta_0(t) = e^{-i\xi t}\Big|_{t=0} = 1$$

(*) Fourier transform of a measure: μ on $\mathbb{R}^d$

$$\xi \in \mathbb{R}^d \mapsto \hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, t\rangle} d\mu(t)$$

if we show $|\hat{\mu}(\xi)| \xrightarrow[|\xi| \to +\infty]{} 0$ then we know μ does not give mass to points.

if $d > 1$ we actually want to prove that μ does not give mass to strict subspaces.

the same condition works (but needs more work because the indicator of a subspace is not a probabilistic measure)

Prop: Let μ be a proba measure on $\mathbb{R}^d$ with support in a subspace $V \subset \mathbb{R}^d$.
Assume $\exists \varepsilon > 0$ s.t. $\forall \xi \in V$
$$|\hat{\mu}(\xi)| \ll \min(1, \|\xi\|^{-\varepsilon})$$
then μ does not assign mass to strict affine subspaces of V.

Now compute
$$\hat{\mu}_T(\xi) = \int e^{-i\langle \xi, t\rangle} d\mu_T(t)$$
$$= \int_A \exp\Big(-i \langle \xi, \underbrace{2\mathrm{Re}\big(\sum_\gamma c_\gamma e^{i2\pi a}\big)}_{\text{finite}} \rangle\Big) d\mathrm{Haar}_A(a)$$

$d=1$, and linear indep
$$= \prod_\gamma \underbrace{\int_0^1 \exp\big(-i\xi\, \underbrace{2\mathrm{Re}(c_\gamma e^{i2\pi\theta})}_{|c_\gamma|\cos(\theta+\varphi)}\big) d\theta}_{\text{Bessel } J_0(|2\xi c_\gamma|) \lesssim \min(1, \frac{1}{\sqrt{|2\xi c_\gamma|}})}$$

$d=1$ and finite number of γ's
$$= \int_A \exp\Big(-i\xi\, \underbrace{2\mathrm{Re}\big(\sum_\gamma c_\gamma e^{i2\pi a}\big)}_{\text{linear combination of cos with phases}}\Big) d\mathrm{Haar}_A(a)$$

analytic function on A
↳ non locally constant
↓
one non vanishing derivative of order k

then $\ll \min(1, |\xi|^{-1/k})$

$\hat{\mu}_T \to \hat{\mu}$ pointwise

$d \geq 2$ and finite number of γ's, some separate by coordinate $\xi_1 f_1 + \xi_2 f_2 + \dots$ (Harmonic analysis) oscillatory integrals.

$d=1$ and infinite number of γ's? ↑ does not work

assume the set of γ's contains a finite set that is linearly indep of the rest

then $\mu = \mu_\Gamma \circledast \mu'$ (convolution) Γ Fourier transform is the product.