Friday, February 17

no classes next week!

Recall: if Θ = supremum of real parts of zeros of $\zeta(s)$ (so that $\frac{1}{2} \le \Theta \le 1$, and RH $\Leftrightarrow \Theta = \frac{1}{2}$), then

$$\psi(x) - x \ll x^{\Theta} \log^2 x.$$

What about corresponding lower bounds?

Recall: "Landau's theorem" (Theorem 1.7 in MV): Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ have abscissa of convergence σ_c. If $a_n \ge 0$ for all sufficiently large n, then $\alpha(s)$ has a singularity at $s = \sigma_c$.

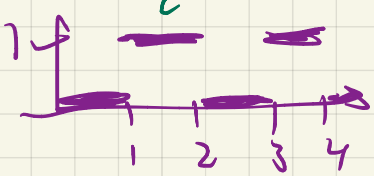
An analogy:

Lemma 15.1: Let $A(x)$ be a bounded, Riemann integrable function, and suppose that $A(x) \ge 0$ for $x \ge x_0$. Let σ_c be the infimum of $\sigma \in \mathbb{R}$ such that $\int_1^{\infty} A(x) x^{-s}\,dx$ converges. Then

$$F(s) = \int_1^{\infty} A(x) x^{-s}\,dx$$

(is analytic in $\{\sigma > \sigma_c\}$ and) has a singularity at $s = \sigma_c$.

Example: Let $S(x) =$



and $A(x) = \frac{S(x)}{x}$.

$$\int_{1^-}^{\infty} x^{-s}\,dS(x) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = \zeta(s)(1-2^{1-s})$$

$$\Rightarrow \frac{1}{s}\zeta(s)(1-2^{1-s}) = \cancel{s}\int \underbrace{\cancel{S(x)}}_{A(x)} x^{-s\cancel{-1}}\,dx$$

for $\sigma > 0$

General phenomenon:

if $S(x) = \sum_{n \le x} a_n$ and $A(x) = \frac{S(x)}{x}$

then $$\frac{1}{s}\alpha(s) = \int_1^\infty A(x)\, x^{-s}\, dx$$

where $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

Typical usage of Landau's theorem:

- If $A(x)$ is eventually nonnegative, then the rightmost singularity of $F(s)$ is on the real axis.
- Contrapositive: if the rightmost singularity of $F(s)$ is not on the real axis, then $A(x)$ is not eventually nonnegative (or eventually nonpositive).

Thus $A(x)$ changes sign infinitely often.

"Rightmost" means "largest real part".

Recall:

$$\psi(x) = \sum_{n \le x} \Lambda(n) \qquad \theta(x) = \sum_{p \le x} \log p$$

$$\Pi(x) = \sum_{2 \le n \le x} \frac{\Lambda(n)}{\log n} \qquad \pi(x) = \sum_{p \le x} 1$$

Let Θ be the supremum of real parts of zeros of $\zeta(s)$.

Theorem 15.2: For every $\varepsilon > 0$,

$$\psi(x) - x = \Omega_{\pm}\left(x^{\Theta - \varepsilon}\right).$$

(In other words,

$$\limsup_{x \to \infty} \frac{\psi(x) - x}{x^{\Theta - \varepsilon}} > 0 \quad \text{and} \quad \liminf_{x \to \infty} \frac{\psi(x) - x}{x^{\Theta - \varepsilon}} < 0.)$$

And $\Pi(x) - \mathrm{li}(x) = \Omega_{\pm}\left(x^{\Theta - \varepsilon}\right)$.

Proof: We know $-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \psi(x)\, x^{-s-1}\,dx$

and so

$$\int_1^\infty \left(\psi(x) - x - x^{\Theta-\varepsilon}\right) x^{-s-1}\,dx$$

$$= \underbrace{-\frac{1}{s}\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1}}_{\text{(cancellation)}} - \frac{1}{s-\Theta+\varepsilon}.$$

- no singularity at $s=1$

$\Rightarrow$ largest real singularity is at $s = \Theta - \varepsilon$.

- $\zeta(s)$ has zeros with real part larger than $\Theta - \varepsilon$ (by def'n of Θ)

so the RHS has singularities to the right of $\Theta - \varepsilon$.

By Landau's theorem $\psi(x) - x - x^{\Theta-\varepsilon} > 0$ infinitely often. (proves Ω_+)

Similarly, looking at $\psi(x) - x + x^{\Theta-\varepsilon}$ establishes the Ω_- half. //

Similarly, one can show

$$s\int_2^\infty li(x)\, x^{-s-1}\,dx = -\log(s-1) + r(s)$$

where $r(s)$ is entire; and then

$$\int_2^\infty \left(x^{\Theta-\varepsilon} - (\Pi(x) - li(x))\right) x^{-s-1}\,dx$$

$$= \frac{1}{s-\Theta+\varepsilon} - \frac{1}{s}\log\left(\zeta(s)(s-1)\right) + \frac{r(s)}{s}$$

and examine singularities ....

Recall: $\theta(x) = \psi(x) - x^{\frac{1}{2}} + O\left(x^{\frac{1}{2}} \exp(-c\sqrt{\log x})\right)$

$$\pi(x) = \Pi(x) - \frac{x^{1/2}}{\log x} + O\left(\frac{x^{1/2}}{\log^2 x}\right)$$

So: • if $\Theta > \frac{1}{2}$, then

$$\theta(x) - x = \Omega_\pm\left(x^{\Theta-\varepsilon}\right)$$

$$\pi(x) - li(x) = \text{some } \Omega_\pm$$

- If $\Theta < \frac{1}{2}$, then we only know

$$\theta(x) - x = \Omega_-(x^{\Theta - \varepsilon})$$

$$\pi(x) - li(x) = \text{some } \Omega_-.$$

Theorem 15.3: Suppose $\zeta(\rho) = 0$ with $\mathrm{Re}\,\rho = \Theta$. ("Supremum is Attained", SA). Then

$$\limsup_{x \to \infty} \frac{\psi(x) - x}{x^{\Theta}} \geq \frac{1}{|\rho|}.$$

$$\liminf_{x \to \infty} \frac{\psi(x) - x}{x^{\Theta}} \leq -\frac{1}{|\rho|}.$$

In particular, unconditionally

$$\psi(x) - x = \Omega_\pm(\sqrt{x})$$

$$\Pi(x) - li(x) = \cancel{\Omega_\pm(\sqrt{x})}\ \Omega_\pm\left(\frac{\sqrt{x}}{\log x}\right).$$

But only Ω for $\theta(x) - x$ and $\pi(x) - li(x)$.

Littlewood (1918):

$$\psi(x) - x = \Omega_\pm(\sqrt{x} \log\log\log x).$$

$$\Pi(x) - li(x) = \Omega_\pm\left(\sqrt{x}\, \frac{\log\log\log x}{\log x}\right).$$

$$\Downarrow$$

$$\theta(x) - x = \Omega_\pm(\sqrt{x} \log\log\log x)$$

$$\pi(x) - li(x) = \Omega_\pm\left(\sqrt{x}\, \frac{\log\log\log x}{\log x}\right)$$

In particular,

$\pi(x) > li(x)$ infinitely often.

Montgomery's Conjecture:

right maximal order of $\psi(x) - x$ should be $\sqrt{x}(\log\log\log x)^2$ $\left(\frac{1}{2\pi}?\right)$

[Added after class]

We can use the same techniques to get oscillation theorems for things like

(1) $\psi(x;q,a) - \frac{x}{\phi(q)}$

$$-\frac{1}{\phi(q)} \sum_{\chi \pmod q}' \bar{\chi}(a) \sum_{L(\rho,\chi)=0} \frac{x^\rho}{\rho} + \text{small}$$

or $\psi(x;q,a) - \psi(x;q,b)$

(2) $$= \frac{1}{\phi(q)} \sum_{\chi \pmod q}' \left(\bar{\chi}(b) - \bar{\chi}(a)\right) \sum_{\rho}' \frac{x^\rho}{\rho} + \text{small}.$$

Notice (1) depends on the poles of $\sum_{\chi}' \bar{\chi}(a) \frac{L'}{L}(s,\chi)$ and (2) depends on the zeros of $\sum_{\chi}' \left(\bar{\chi}(a) - \bar{\chi}(b)\right) \frac{L'}{L}(s,\chi)$;

both sets of zeros are subsets of the set of zeros of $\prod_{\chi \pmod q} L(s,\chi)$, but there might be cancellation in (2) (when $\chi(a)=\chi(b)$) or (1)/(2) (when two $L(s,\chi)$ share a zero).

So "(1)" could depend on a and/or b in principle.