The sum of the Möbius function

Nathan Ng

University of Lethbridge

Analytic Number Theory 2 (UBC)
PIMS network course (instructor G. Martin)
Guest Lecture (online)
Feb. 27, 2023

Table of Contents

Table of Contents

Notation

- Let $g(x) > 0$.
- $f(x) = O(g(x))$ and $f(x) \ll g(x)$ mean that there exist positive constants C and x_0 such that
$$|f(x)| \leq Cg(x) \text{ for } x \geq x_0.$$
- $f(x) \asymp g(x)$ means $f(x) \ll g(x)$ and $g(x) \ll f(x)$.
- $f(x) = \Omega_+(g(x))$ means
$$\limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0.$$
- $f(x) = \Omega_-(g(x))$ means
$$\liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$
- $f(x) \sim g(x)$ means $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.
- $f(x) = o(g(x))$ means $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

The Classical Summatory functions

- The ψ function of prime number theory.

$$\psi(x) = \sum_{n \le x} \Lambda(n) \text{ where } \Lambda(n) = \begin{cases} \log p & \text{if } n = p^j, \\ 0 & \text{else .} \end{cases}$$

- The sum of the Möbius function.

$$M(x) = \sum_{n \le x} \mu(n) \text{ where } \mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ is squarefree}, \\ 0 & \text{else.} \end{cases}$$

- The sum of the Liouville function.

$$L(x) = \sum_{n \le x} \lambda(n) \text{ where } \lambda(n) = (-1)^{\Omega(n)} = (-1)^{a_1 + \cdots a_k} \text{ and } n = p_1^{a_1} \cdots p_k^{a_k}.$$

- A weighted version of $L(x)$.

$$T(x) = \sum_{n \le x} \frac{\lambda(n)}{n}$$

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- A weighted version of $L(x)$.

$$T(x) = \sum_{n \le x} \frac{\lambda(n)}{n}$$

The prime number theorem

- The prime number theorem (PNT) is the assertion

$$\pi(x) = \#\{p \le x \mid p \text{ prime } \} \sim \int_2^x \frac{dt}{\log t}.$$

- By an exercise (partial summation), the PNT is equivalent to

$$\psi(x) \sim x.$$

- The Riemann hypothesis is equivalent to

$$\psi(x) = x + O(\sqrt{x}\log^2 x).$$

- The prime number theorem is also equivalent to the statement

$$M(x) = o(x).$$

Table of Contents

The size of $M(x)$. Computations of Deleglise and Rivat

n	10	11	12	13	14	15
$M(1\times 10^n)$	−33722	−87856	62366	599582	−875575	−3216373
$M(2\times 10^n)$	48723	−19075	−308413	127543	2639241	1011871
$M(3\times 10^n)$	42411	133609	190563	−759205	−2344314	5334755
$M(4\times 10^n)$	−25295	202631	174209	−403700	−3810264	−6036592
$M(5\times 10^n)$	54591	56804	−435920	−320046	4865646	11792892
$M(6\times 10^n)$	−56841	−43099	268107	1101442	−4004298	−14685733
$M(7\times 10^n)$	7917	111011	−4252	−2877017	−2605256	4195668
$M(8\times 10^n)$	−1428	−268434	−438208	−99222	3425855	6528429
$M(9\times 10^n)$	−5554	10991	290186	1164981	7542952	−12589671

$\widetilde{M}(x) = |M(x)|/\sqrt{x}$.

n	10	11	12	13	14	15
$\widetilde{M}(1\times 10^n)$	0.3372	0.2778	0.0624	0.1896	0.0876	0.1017
$\widetilde{M}(2\times 10^n)$	0.3445	0.0427	0.2181	0.0285	0.1866	0.0226
$\widetilde{M}(3\times 10^n)$	0.2449	0.2439	0.1100	0.1386	0.1353	0.0974
$\widetilde{M}(4\times 10^n)$	0.1265	0.3204	0.0871	0.0638	0.1905	0.0954
$\widetilde{M}(5\times 10^n)$	0.2441	0.0803	0.1949	0.0453	0.2176	0.1668
$\widetilde{M}(6\times 10^n)$	0.2321	0.0556	0.1095	0.1422	0.1635	0.1896
$\widetilde{M}(7\times 10^n)$	0.0299	0.1327	0.0016	0.3439	0.0985	0.0501
$\widetilde{M}(8\times 10^n)$	0.0050	0.3001	0.1549	0.0111	0.1211	0.0730
$\widetilde{M}(9\times 10^n)$	0.0185	0.0116	0.0967	0.1228	0.2514	0.1327

Known results.

Theorem

Unconditionally, $M(x) = O\left(x \exp\left(-c_1 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\right)\right)$ *for* $c_1 > 0$ [1]

RH implies

$$M(x) = O\left(x^{\frac{1}{2}} \exp\left((\log x)^{\frac{1}{2}} (\log\log x)^{\frac{7}{8}}\right)\right)$$ [2]

Theorem (Selection of explicit results [3])

$$\begin{aligned}
\textit{von Sterneck, 1898} \quad |M(x)| &\leq \frac{x}{9} + 8 \quad \text{for} \quad x \geq 0, \\
\textit{Cohen, Dress, El Marraki, 2007} \quad |M(x)| &< \frac{x}{4\,345} \quad \text{for} \quad x \geq 2\,160\,535, \\
\textit{Schoenfeld, 1968} \quad |M(x)| &< \frac{0.55x}{(\log x)^{2/3}} \quad \text{for} \quad x > 1, \\
\textit{Ramaré, 2013} \quad |M(x)| &< \frac{0.013x}{\log x} \quad \text{for} \quad x \geq 1078853, \\
\textit{Hurst, 2018} \quad |M(x)| &\leq 0.571\sqrt{x} \quad \text{for} \quad 33 \leq x \leq 10^{16}.
\end{aligned}$$

[1] see Ivić's book, Theorem 12.7, pp. 309-315.

[2] Soundararjan, 2009 and Bui-Florea 2023

[3] See article of Lee-Leong for many references.

Omega results

$f(x) = \Omega_+(g(x))$ means $\limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0$.

Theorem (Omega results [4])

(i)

$$M(x) = \Omega_\pm(x^{\frac{1}{2}}) \ .$$

(ii) If $\zeta(s)$ has a multiple zero of of order $m \geq 2$

$$M(x) = \Omega_\pm(x^{\frac{1}{2}}(\log x)^{m-1}) \ .$$

(iii) If RH is false, then

$$M(x) = \Omega_\pm(x^{\theta-\delta})$$

where

$$\theta = \sup_{\rho, \zeta(\rho)=0} \mathrm{Re}(\rho)$$

and δ is any positive constant.

This shows that to disprove the Mertens conjecture, one can assume RH is true and all zeros of $\zeta(s)$ are simple.

[4] See pp.467-470 of Montgomery-Vaughan's book.

Table of Contents

The Mertens Conjecture

Conjecture (Mertens Conjecture)

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$$|M(x)| \leq \sqrt{x}.$$

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$$\int_1^X \left(\frac{M(x)}{x} \right)^2 dx \ll \log X.$$

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Exercise. Each of these conjectures implies the Riemann Hypothesis.

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx.$$

Why the Mertens Conjecture should be false.

- We expect that for every $\varepsilon > 0$

$$M(x) = O(x^{\frac{1}{2}+\varepsilon})$$

- What is the smallest function $f(x) \to \infty$ such that $M(x) = O(x^{\frac{1}{2}} f(x))$.

Conjecture (LI: Linear Independence Conjecture)

The imaginary parts $\gamma_1, \gamma_2, \ldots$ of the (distinct) zeros of $\zeta(s)$ above the real axis are connected by no relation of the type

$$\sum_{n=1}^{N} c_n \gamma_n = 0 \, (c_n \text{ integers not all zero}),$$

Theorem (Ingham)

Assume LI is true. Then

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} = \infty, \ \liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} = -\infty,$$

$$\limsup_{x\to\infty} \frac{L(x)}{\sqrt{x}} = \infty, \ \liminf_{x\to\infty} \frac{L(x)}{\sqrt{x}} = -\infty.$$

Disproving the Mertens Conjecture.

Let

$$A = \limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} \text{ and } B = \liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}}.$$

author	year	A	B
Jurkat	1973		< -0.5054
Spira	1966	≥ 0.5355	≤ -0.6027
Jurkat-Peyerimhoff	1976	≥ 0.779	≤ -0.638
Te Riele	1979	≥ 0.860	≤ -0.843
Odlyzko-te-Riele	1985	≥ 1.06	≤ -1.009
Kotnik-te-Riele	2003	≥ 1.279	≤ -1.218
Best-Trudgian	2015	≥ 1.6383	≤ -1.6383
Hurst	2018	≥ 1.837625	≤ -1.837625

Table of Contents

Discrete moments of the Riemann zeta function.

Conjecture (SZ: Simple zeros Conjecture)

All zeros of the Riemann zeta function are simple.

$$\text{Note that } \rho \text{ is simple } \iff \zeta'(\rho) \neq 0.$$

- The theory of $M(x)$ is intimately related to negative moments of $|\zeta'(\rho)|$.

$$J_{-k}(T) = \sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^{2k}} \text{ for } k \in \mathbb{R}.$$

Conjecture (Gonek-Hejhal, 1989)

For $k \in [0, \frac{3}{2})$,

$$J_{-k}(T) \asymp T(\log T)^{(k-1)^2}.$$

Lower bounds for discrete moments

Conjecture (Gonek,1989 and Hughes-Keating-O'Connell, 200*)

$$J_{-1}(T) \sim \frac{3}{\pi^3} T \text{ and } J_{-\frac{1}{2}}(T) \sim \alpha T(\log T)^{\frac{1}{4}}$$

for a precise constant α.

Theorem (Milinovich-N., 2010)

Assume the Riemann Hypothesis and the zeros of $\zeta(s)$ *are simple. Then for every* $\varepsilon > 0$,

$$J_{-1}(T) \geq \Big(\frac{3}{2\pi^3} - \varepsilon\Big) T,$$

for T *sufficiently large.*

Theorem (Heap-Li-Zhao, 2022)

Assume the Riemann Hypothesis and the zeros of $\zeta(s)$ *are simple. then*

$$J_{-\frac{1}{2}}(T) \gg T(\log T)^{\frac{1}{4}}.$$

Note: Currently there are no non-trivial upper bounds for $J_{-1}(T)$ and $J_{-\frac{1}{2}}(T)$. In the study of $M(x)$ we must assume some average bound for $\frac{1}{|\zeta'(\rho)|}$. See recent work of Bui-Florea.

Table of Contents

An explicit formula for $M(x)$.

Exercise: Use Perron's formula to show for $1 \le T \le x$:

$$M(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} ds + O\left(\frac{x \log x}{T}\right) \text{ where } c = 1 + \frac{1}{\log x}.$$

Comparison to PNT:

$$\psi(x) = -\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \frac{\zeta'(s)}{\zeta(s)} ds + O\left(\frac{x(\log x)^2}{T}\right) \text{ where } c = 1 + \frac{1}{\log x}.$$

Move contour left to $\mathfrak{Re}(s) = -\frac{1}{10}$

function	integrand	pole	residue
$\psi(x)$	$-\frac{\zeta'(s)x^s}{\zeta(s)s}$	$s = 1$	x
$\psi(x)$	$-\frac{\zeta'(s)x^s}{\zeta(s)s}$	$s = \rho$	$-\frac{x^\rho}{\rho}$
$M(x)$	$\frac{x^s}{\zeta(s)s}$	$s = \rho$, ρ simple zero	$\frac{x^\rho}{\rho\zeta'(\rho)}$

Exercise. Compute residue$\left(\frac{x^s}{\zeta(s)s}, s = \rho\right)$ assuming ρ is a zero of order m.

Lemma (Special sequence $\mathcal{T}$ [5])

Assume RH. Let $\epsilon > 0$. There exists a sequence of numbers $\mathcal{T} = \{T_n\}_{n=0}^{\infty}$ which satisfies

$$n \leq T_n \leq n+1 \text{ and } \frac{1}{\zeta(\sigma + iT)} = O(T^{\epsilon})$$

for all $-1 \leq \sigma \leq 2$.

[5] See Lemma 3 of Ng, PLMS, 2004

[6] See Lemma 4 of Ng, PLMS, 2004

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for all $-1 \le \sigma \le 2$.

Theorem ($M(x)$ explicit formula [6])

Let $\epsilon > 0$. Assume the Riemann hypothesis and that all zeros of $\zeta(s)$ are simple. For $x \ge 2$ and $T \in \mathcal{T}$

$$M(x) = \sum_{|\gamma|<T} \frac{x^{\rho}}{\rho\zeta'(\rho)} + O\Big(\frac{x\log x}{T} + \frac{x}{T^{1-\epsilon}\log x} + 1\Big).$$

Let $\rho = \frac{1}{2} + i\gamma$ be a non-trivial zero.

$$M(x)x^{-\frac{1}{2}} = \sum_{|\gamma|<T} \frac{x^{i\gamma}}{\rho\zeta'(\rho)} + O\Big(\frac{x^{\frac{1}{2}}\log x}{T} + \frac{x^{\frac{1}{2}}}{T^{1-\epsilon}\log x} + x^{-\frac{1}{2}}\Big).$$

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For an appropriate $T = T(x) \geq Cx^{\frac{1}{2}} \log x$

$$M(x)x^{-\frac{1}{2}} = \sum_{|\gamma|<T} \frac{x^{i\gamma}}{\rho\zeta'(\rho)} + o(1).$$

For an appropriate $T = T(x) \geq Cx^{\frac{1}{2}} \log x$

$$M(x)x^{-\frac{1}{2}} = \sum_{|\gamma|<T} \frac{x^{i\gamma}}{\rho\zeta'(\rho)} + o(1).$$

Pair conjugate zeros in the sum $\rho = \frac{1}{2} + i\gamma$ and $\overline{\rho} = \frac{1}{2} - i\gamma$

$$\begin{aligned}
\frac{x^{i\gamma}}{\rho\zeta'(\rho)} + \frac{x^{-i\gamma}}{\overline{\rho}\zeta'(\overline{\rho})} &= 2\mathfrak{Re}\Big(\frac{x^{i\gamma}}{\rho\zeta'(\rho)}\Big) \text{ since } \zeta'(\overline{\rho}) = \overline{\zeta'(\rho)} \\
&= 2\mathfrak{Re}\Big(\frac{x^{i\gamma}}{|\rho\zeta'(\rho)|e^{i\arg(\rho\zeta'(\rho))}}\Big) \\
&= \frac{2}{|\rho\zeta'(\rho)|}\mathfrak{Re}\Big(e^{i\gamma\log x + i\beta_\gamma}\Big) \text{ where } \beta_\gamma = -\arg(\rho\zeta'(\rho))
\end{aligned}$$

$$M(x)x^{-\frac{1}{2}} = 2\mathfrak{Re}\Big(\sum_{0<\gamma<T} \frac{e^{i(\gamma\log x + \beta_\gamma)}}{|\rho\zeta'(\rho)|}\Big) + o(1)$$

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$$M(x)x^{-\frac{1}{2}} = 2\mathfrak{Re}\Big(\sum_{0<\gamma<T} \frac{e^{i(\gamma\log x + \beta_\gamma)}}{|\rho\zeta'(\rho)|}\Big) + o(1)$$

Variable change $x = e^y$

$$M(e^y)e^{-\frac{y}{2}} = 2\mathfrak{Re}\Big(\sum_{0<\gamma<T_y} \frac{e^{i(\gamma y+\beta_\gamma)}}{|\rho\zeta'(\rho)|}\Big) + o(1) = 2\mathfrak{Re}\Big(\sum_{j=1}^{N} \frac{e^{2\pi i(\frac{\gamma_j}{2\pi}y+\widetilde{\beta}_j)}}{|\rho_j\zeta'(\rho_j)|}\Big) + o(1)$$

where non-trivial zeros have been labelled as $\rho_j = \frac{1}{2} + i\gamma_j$ with

$$\gamma_1 \le \gamma_2 \le \cdots \le \gamma_j \le \gamma_{j+1} \le \cdots .$$

[7] See. Titchmarsh, Theorem 8.3, p.185 and Ivić, Lemma 9.3, p.233

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Theorem (Kronecker [7])

Let $a_1, \ldots, a_N$ be linearly independent over the rational numbers.That is, there is no linear relation $\lambda_1 a_1 + \cdots + \lambda_N a_N = 0$ where $\lambda_1, \ldots, \lambda_N$ are integers (not all zero). Let $b_1, \ldots, b_N$ be any real numbers, and $\varepsilon \in (0,1)$. Then we can find a number y and integers $x_1, \ldots, x_N$ such that

$$|a_j y - b_j - x_j| \le \varepsilon \ \ (j = 1, \ldots, N).$$

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Variable change $x = e^y$

$$M(e^y)e^{-\frac{y}{2}} = 2\Re\mathfrak{e}\Big(\sum_{0<\gamma<T_y} \frac{e^{i(\gamma y+\beta_\gamma)}}{|\rho\zeta'(\rho)|}\Big) + o(1) = 2\Re\mathfrak{e}\Big(\sum_{j=1}^{N} \frac{e^{2\pi i(\frac{\gamma_j}{2\pi}y+\widetilde{\beta}_j)}}{|\rho_j\zeta'(\rho_j)|}\Big) + o(1)$$

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$$|a_j y - b_j - x_j| \le \varepsilon \ \ (j = 1, \ldots, N).$$

Applying with $a_j = \frac{\gamma_j}{2\pi}$ and $b_j = -\widetilde{\beta}_j$ for $j = 1, \ldots, N$, we see that there exists y

$$\frac{\gamma_j}{2\pi}y + \widetilde{\beta}_j \text{ is close to an integer } \implies e^{2\pi i(\frac{\gamma_j}{2\pi}y+\widetilde{\beta}_j)} \approx 1.$$

[7] See. Titchmarsh, Theorem 8.3, p.185 and Ivić, Lemma 9.3, p.233

$$M(e^y)e^{-\frac{y}{2}} \approx 2\mathfrak{Re}\Big(\sum_{j=1}^{N} \frac{1}{|\rho_j \zeta'(\rho_j)|}\Big) + o(1)$$

Assuming RH and SZ (all zeros are simple), Ingham showed

$$\sum_{j=1}^{\infty} \frac{1}{|\rho_j \zeta'(\rho_j)|} = \infty.$$

Note that the Gonek-Hejhal conjecture implies

$$\sum_{0<\gamma<T} \frac{1}{|\zeta'(\rho)|} \asymp T(\log T)^{\frac{1}{4}}. \qquad (1)$$

Exercise. Use partial summation to show that (1) implies

$$\sum_{0<\gamma<T} \frac{1}{|\rho\zeta'(\rho)|} \asymp (\log T)^{\frac{5}{4}}.$$

Thus we expect for this value of y:

$$M(e^y)e^{-\frac{y}{2}} \approx (\log N)^{\frac{5}{4}}.$$

Theorem

The Riemann Hypothesis and $J_{-1}(T) \ll T$ imply:
(i)

$$M(x) \ll x^{\frac{1}{2}} (\log x)^{\frac{3}{2}} \ ,$$

(ii) and the weak Mertens conjecture is true

$$\int_2^X \left(\frac{M(x)}{x} \right)^2 \ dx \ll \log X \ .$$

- Assuming the bound $J_{-1/2}(T) \ll T(\log T)^{\frac{1}{4}}$ (i) becomes

$$M(x) \ll x^{\frac{1}{2}} (\log x)^{\frac{5}{4}}.$$

- Compare to von Koch's RH implies

$$\psi(x) = x + O(\sqrt{x}(\log x)^2).$$

Table of Contents

Heath-Brown 1992:
"It appears to be an open question whether $x^{-\frac{1}{2}}M(x)$ has a distribution function. To prove this one would want to assume the Riemann Hypothesis and the simplicity of the zeros, and perhaps also a growth condition on $M(x)$."

Theorem (N., 2004, PLMS)

Assume RH and $J_{-1}(T) \ll T$. Then $e^{-\frac{y}{2}}M(e^y)$ has a limiting distribution ν on $\mathbb{R}$, that is,

$$\lim_{Y\to\infty} \frac{1}{Y}\int_0^Y f(e^{-\frac{y}{2}}M(e^y))\,dy = \int_{-\infty}^{\infty} f(x)\,d\nu(x) \tag{2}$$

for all bounded Lipschitz continuous functions f on $\mathbb{R}$.

- Techniques of Rubinstein-Sarnak, Cramer.
- Kronecker-Weyl equidistribution theorem.
  (the ray $y(\gamma_1, \ldots, \gamma_N)$ is uniformly distributed in its closure within $\mathbb{T}^N$)
- ν is a probability measure which reveals behaviour of $M(x)$.
- Approximation argument gives f bounded, continuous functions.

Suppose the above theorem remains valid for indicator functions. Let $f = 1_{[a,b]}$ where

$$1_{[a,b]}(x) = \begin{cases} 1 \text{ if } x \in [a,b] \\ 0 \text{ else} \end{cases} .$$

With the above choice of $f(x)$ the theorem translates to

$$\lim_{Y \to \infty} \frac{1}{Y} \mathrm{meas}\{y \in [0,Y] \mid a \leq \frac{M(e^y)}{e^{\frac{y}{2}}} \leq b\} = \nu([a,b]). \tag{3}$$

- If ν is an absolutely continuous measure, then this identity would be true.
- The measure ν tells us how often the scaled version of $M(x)/\sqrt{x}$ lies between a and b. (eg. take $a=-1$ and $b=1$)

A random model for $M(x)$.

Recall

$$M(e^y)e^{-\frac{y}{2}} \sim 2\mathfrak{Re}\Bigg(\sum_{j=1}^{N} \frac{e^{i(\gamma_j y+\beta_{\gamma_j})}}{|\rho_j \zeta'(\rho_j)|}\Bigg) \text{ for some } N = N_y.$$

A random model for $M(x)$.

Recall

$$M(e^y)e^{-\frac{y}{2}} \sim 2\mathfrak{Re}\Big(\sum_{j=1}^{N} \frac{e^{i(\gamma_j y+\beta_{\gamma_j})}}{|\rho_j \zeta'(\rho_j)|}\Big) \text{ for some } N = N_y.$$

Assuming LI this behaves like

$$X(\underline{\theta}) = \sum_{n=1}^{\infty} r_n \cos(2\pi\theta_n) \text{ where } r_n = \frac{2}{|\rho_n \zeta'(\rho_n)|}. \tag{4}$$

where $\underline{\theta} = (\theta_1, \theta_2, \ldots,) \in \mathbb{T}^\infty$ (i.i.d. variables in $[0, 1]$).

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On RH, LI, and $J_{-1}(T) \ll T$, we have
$$\nu(B) = P_X(B).$$

- Using methods from probability, we can show there exist $c_1, c_2 > 0$

$$\exp(-\exp(c_1 V^{\frac{4}{5}})) \ll P_X([V,\infty)) \ll \exp(-\exp(c_2 V^{\frac{4}{5}})). \tag{5}$$

- A heurstic argument suggests

$$\left(\frac{1}{c_1}\right)^{\frac{5}{4}} \leq \limsup_{y\to\infty} \frac{M(e^y)}{e^{\frac{y}{2}}(\log\log y)^{\frac{5}{4}}} \leq \left(\frac{1}{c_2}\right)^{\frac{5}{4}} . \tag{6}$$

Conjecture (Gonek)

There exists a number $B > 0$ such that

$$\overline{\lim}_{x\to\infty} \frac{M(x)}{\sqrt{x}(\log\log\log x)^{\frac{5}{4}}} = \pm B . \tag{7}$$

Conjecture (Ng (2012))

The value of B in (7) is $B = \frac{8a}{5}$ where

$$a = \frac{1}{\sqrt{\pi}} e^{3\zeta'(-1) - \frac{11}{12}\log 2} \prod_p \left((1-p^{-1})^{\frac{1}{4}} \sum_{k=0}^{\infty} \left(\frac{\Gamma(k-\frac{1}{2})}{k!\Gamma(-\frac{1}{2})} \right)^2 p^{-k} \right) = 0.16712\ldots[8]$$

[8] This value is based on independent numerical calculations of Harald Helfgott and Michael Rubinstein.

Table of Contents

Polya and Turan problems.

Polya observed that

$$L(x) = \sum_{n \le x} \lambda(n) \le 0 \, (\text{for } 2 \le x \le 250,000)$$

and Turan observed

$$T(x) = \sum_{n \le x} \frac{\lambda(n)}{n} \ge 0$$

for small values of x. Does this persist forever?

Theorem (Haselgrove)

(i) $L(x)$ changes signs infinitely often.
(ii) $T(x)$ changes signs infinitely often,.

$$\begin{aligned} \textit{Tanaka}, 1980, &\quad L(906105257) > 0, \\ \text{Borwein, Ferguson, Mossinghoff, 2008}, &\quad T(72185376951205) < 0. \end{aligned}$$

See article of Humphries for limiting distribution results on $L(x)$ and $T(x)$ and article of Mossinghoff and Trudgian.

Table of Contents

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