Friday, February 3

We're in the middle of proving:

Theorem 11.3 (MV) — "zero-free region" for $L(s,\chi)$.

There exists $c>0$ such that, for any $\chi \pmod q$, the region

$$\left\{ s=\sigma+it : \sigma > 1 - \frac{c}{\log q\tau} \right\}$$

contains no zeros of $L(s,\chi)$, unless χ is quadratic, in which case there might be one real zero $\beta_1 < 1$.

Proof for complex χ used:

- $\mathrm{Re}\left(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right) \geq 0.$

Suppose $\rho_0 = \beta_0 + i\gamma_0$ is a zero of $L(s,\chi)$.

- $\mathrm{Re}\left(-\frac{L'}{L}(1+\delta,\chi_0)\right) \leq \frac{1}{\delta} + O(\log q\gamma_0)$

$\mathrm{Re}\left(-\frac{L'}{L}(1+\delta+i\gamma_0,\chi)\right) \leq -\frac{1}{1+\delta-\beta_0} + O(\quad)$

$\mathrm{Re}\left(-\frac{L'}{L}(1+\delta+2i\gamma_0,\chi^2)\right) \leq O(\log q\gamma_0).$

↳ if $\chi^2 \neq \chi_0$.

So we have "3 votes" for poles, +
"4 votes" for zeros, −

Now consider χ quadratic:

- If $|\gamma_0|$ is not too small, then we're not close to the pole of $L(s,\chi^2) = L(s,\chi_0)$; the argument goes through.

↳ Side note: this is the proof of zero-free region for $\zeta(s)$.

To show $\zeta(s)$ zero-free near real axis:

one can show (sum over nontrivial zeros ρ of $f(s)$)

$$\sum_\rho \mathrm{Re}\frac{1}{\rho} = \sum_\rho \frac{\beta}{\beta^2+\gamma^2} = \frac{c_0}{2} + 1 - \frac{1}{2}\log 4\pi$$

$$\approx 0.0231$$

$\Rightarrow |\gamma| \geq 6.5$

- What if $|\gamma_0|$ is small but nonzero?

Trick: since χ is real,

$L(\beta_0+i\gamma_0,\chi)=0 \Rightarrow L(\beta_0-i\gamma_0,\chi)=0.$

"3 votes for poles" $-$ "$8\,\cancel{4}$ votes for zeros" $+$ "1 vote for pole"

- What if there are two real zeros β_1, β_2?

- same: $3-8+1$ votes. ✓

- but can't rule out one real zero

Side note: the initial inequality used the nonnegative "cosine polynomial"

$$3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geq 0.$$

Good news: if

$$T(\theta) = \sum_{n=0}^{N} a_n \cos(n\theta)$$

is nonnegative, then $\frac{a_1}{a_0}$ can be arbitrarily close to 2:

Fejer kernel

$$\Delta_N(\theta) = 1 + 2\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)\cos 2\pi n\theta = \frac{1}{N}\left(\frac{\sin(\pi N\theta)}{\pi N\theta}\right)^2 \geq 0$$

Bad news: $\frac{a_1}{a_0}$ must be strictly less than 2, since

$$a_0 - \frac{1}{2}a_1 = \frac{1}{2\pi}\int_0^{2\pi} T(\theta)(1-\cos\theta)\,d\theta > 0,$$

Using the approximation

$$\frac{L'}{L}(s,\chi) = \sum_{\rho \text{ near } s}' \frac{1}{s-\rho} + O(\log q\tau),$$

one can now show:

<u>Theorem 11.4 (MV)</u>: Let $\chi \pmod q$ be nonprincipal. Suppose that

$$\sigma > 1 - \frac{c/2}{\log q\tau}.$$

- If $L(s,\chi)$ has no exceptional zero, or if β_1 is an exceptional zero but $|s-\beta_1| \geq \frac{1}{\log q}$, then

$$\frac{L'}{L}(s,\chi) \ll \log q\tau$$

$$|\log L(s,\chi)| \leq \log\log q\tau + O(1)$$

$$\frac{1}{\log q\tau} \ll |L(s,\chi)| \ll \log q\tau.$$

- If $|s-\beta_1| < \frac{1}{\log q}$, then

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + O(\log q\tau)$$

$$|\arg L(s,\chi)| \leq \log\log q\tau + O(1)$$

$$\frac{1}{\log q\tau} \ll \frac{|L(s,\chi)|}{|s-\beta_1|} \ll \log q\tau.$$

Theorem 11.7: If $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ are quadratic characters, not induced by the same primitive character, then $L(s,\chi_1)L(s,\chi_2)$ has at most one (real) zero in

$$\left\{\sigma > 1 - \frac{c}{\log q_1 q_2 \tau}\right\}.$$

Proof: For $\sigma > 1$,

$$-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma,\chi_1) - \frac{L'}{L}(\sigma,\chi_2) - \frac{L'}{L}(\sigma,\chi_1\chi_2)$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma}\left(1+\chi_1(n)+\chi_2(n)+\chi_1\chi_2(n)\right)$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma}(1+\chi_1(n))(1+\chi_2(n)) \geq 0.$$

- ζ has a pole at $s=1$: 1 pole vote
- $L(s,\chi_1)$ and $L(s,\chi_2)$ have zeros: 2 zero votes
- $\chi_1\chi_2 \neq \chi_0$, so no pole votes.

Solve the inequalities ... //

Corollary 11.10: For $Q > 1$,

$$\prod_{q \leq Q} \prod_{\substack{\chi \pmod q \\ \chi \text{ primitive}}} L(s,\chi)$$ has no zeros in $\left\{\sigma > 1 - \frac{c}{\log Q\tau}\right\}$, except possibly one real zero.

Corollary 11.9: For $A > 0$, there exists $c(A) > 0$ such that if $q_1 < q_2$; and $L(\beta_1, \chi_1) = 0$ with $\chi_1 \pmod{q_1}$ and $\beta_1 > 1 - \frac{c(A)}{\log q_1}$; and $L(\beta_2, \chi_2) = 0$ with $\chi_2 \pmod{q_2}$ and $\beta_2 > 1 - \frac{c(A)}{\log q_2}$ ...

then $q_2 > q_1^A$.