Wednesday, February 8

Start looking at $\psi(x,\chi) = \sum_{n \le x} \Lambda(n)\chi(n)$,

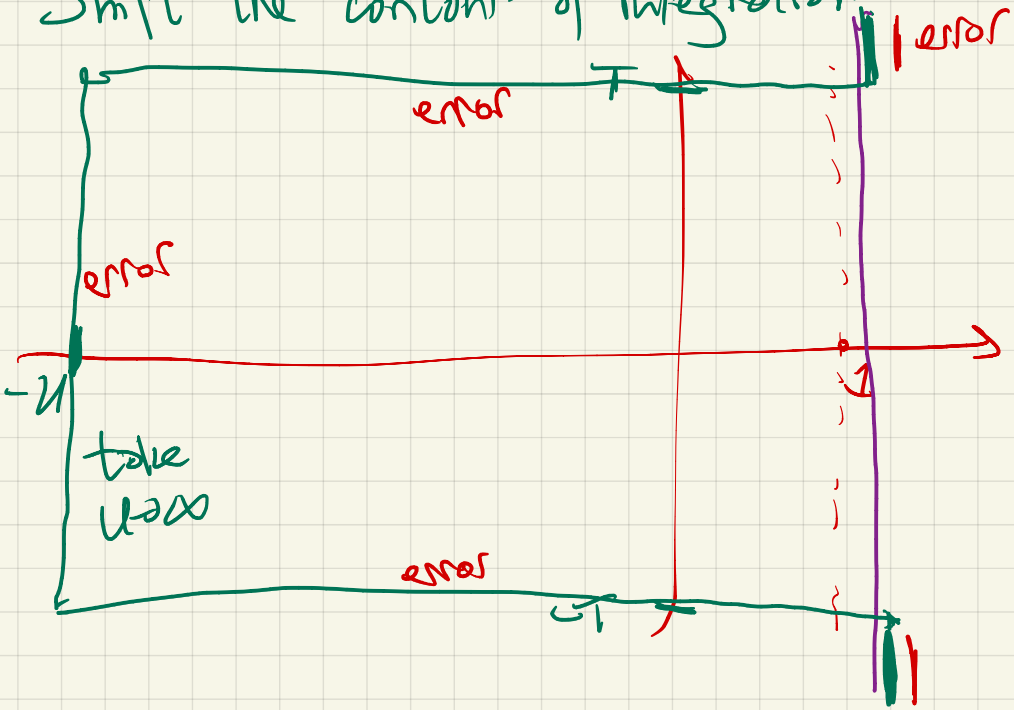
using Perron's formula: for $c > 1$,

$$\psi_0(x,\chi) = \frac{1}{2\pi i} \int_{(c)} -\frac{L'}{L}(s,\chi) \frac{x^s}{s}\, ds$$

where $\int_{(c)}$ means $\int_{c-i\infty}^{c+i\infty}$ and

$$\psi_0(x,\chi) = \frac{1}{2}\left(\psi(x^-,\chi) + \psi(x^+,\chi)\right).$$

Shift the contour of integration:



<u>Theorem 12.10</u> (MV): For $x, T \ge 2$, if $\chi \pmod q$ is primitive then

$$\psi_0(x,\chi) = \left\{x \text{ if } q=1\right\} - \sum_{\substack{\rho \\ |\gamma| \le T}} \frac{x^\rho}{\rho} - \frac{1}{2}\log(x-1) - \frac{\chi(-1)}{2}\log(x+1) + C(\chi),$$

where ρ runs over nontrivial zeros of $L(s,\chi)$ and

$$C(\chi) = \begin{cases} \frac{L'}{L}(1,\bar{\chi}) & \text{if } q>1 \\ + \log\frac{q}{2\pi} - C_0 & \text{(Euler's constant)} \end{cases}$$

$$+ R(x,T,\chi)$$

where $R(x,T,\chi) \ll \log x \cdot \min\left\{1, \frac{x}{T\langle x\rangle}\right\} + \frac{x}{T}(\log(qxT))^2$, where $\langle x\rangle$ = distance from x to the nearest prime power other than x.

Letting $T\to\infty$:

$$\psi_0(x,\chi) = \{x \text{ if } q=1\} - \sum_{\rho} \frac{x^{\rho}}{\rho}$$

$$- \tfrac{1}{2}\log(x-1) - \frac{\chi(-1)}{2}\log(x+1) + C(\chi),$$

"explicit formula" – <u>EXACT</u>.

<u>Side note:</u> what if $\chi \pmod q$ is induced by $\chi^* \pmod{q^*}$?

$$\psi(x,\chi^*) = \sum_{\substack{n\le x\\(n,q^*)=1}}{}' \chi^*(n)\Lambda(n)$$

$$= \underbrace{\sum_{\substack{n\le x\\(n,q)=1}} \chi^*(n)\Lambda(n)}_{\psi(x,\chi)} + \sum_{\substack{n\le x\\(n,q)>1\\(n,q^*)=1}}{}' \chi^*(n)\Lambda(n),$$

Hence $\psi(x,\chi^*) - \psi(x,\chi) =$

$$= \sum_{\substack{n\le x\\(n,q)>1\\(n,q^*)=1}} \chi^*(n)\Lambda(n)$$

$$\ll \sum_{\substack{n\le x\\(n,q)>1}} \Lambda(n) = \sum_{p\mid q}\log p \sum_{\substack{r\ge 1\\p^r\le x}} 1$$

$$\le \sum_{p\mid q}\log p \left\lfloor \frac{\log x}{\log p}\right\rfloor$$

$$\ll \sum_{p\mid q}\log x \ll \log x\cdot\log q.$$

So for all $\chi \pmod q$,

$$\psi(x,\chi) = \{x \text{ if } \chi=\chi_0\} - \sum_{\rho}\frac{x^{\rho}}{\rho} + O\left(\log x\cdot\log q + \left|\frac{L'}{L}(1,\chi)\right|\right)$$

We've seen

$$\psi(x;q,a) = \sum_{\substack{n\le x \\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(n)$$

$$= \frac{1}{\phi(q)} \sum_{\chi \ (\mathrm{mod}\ q)} \overline{\chi}(a)\,\psi(x,\chi)$$

$$= \frac{x}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\ q)} \overline{\chi}(a) \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ \text{nontrivial}}} \frac{x^\rho}{\rho}$$

$$+ O\Bigg(\log q \cdot \log x + \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi\ne\chi_0}} \left|\frac{L'}{L}(1,\chi)\right|\Bigg)$$

Detour to $\psi(x) = \sum_{n\le x} \Lambda(n)$

and relate to $\theta(x) = \sum_{p\le x} \log p$, $\pi(x) = \sum_{p\le x} 1$.

Note that $\psi(x) = \sum_{p^r\le x} \log p$

$$= \sum_{p\le x} \log p + \sum_{p\le\sqrt{x}} \log p + \sum_{p\le\sqrt[3]{x}} \log p + \cdots$$

$$\psi(x) = \theta(x) + \theta(x^{\frac12}) + \theta(x^{\frac13}) + \cdots$$

$$= \sum_{r=1}^{\infty} \theta(x^{1/r}).$$

By "Möbius inversion",

$$\theta(x) = \sum_{r=1}^{\infty} \mu(r)\,\psi(x^{1/r})$$

$$= \psi(x) - \psi(x^{\frac12}) - \psi(x^{\frac13}) - \psi(x^{\frac15}) + \psi(x^{1/6}) + \cdots$$

$$= \psi(x) - (x^{\frac12} + \text{error}) + (\text{error})$$

So

$$\frac{\theta(x) - x}{\sqrt{x}} = \frac{\psi(x) - x}{\sqrt{x}} - 1 + o(1).$$

$$-\sum_{\rho} \frac{x^{\rho-\frac12}}{\rho}$$

from squares of primes

To go from $\theta(x)$ to $\pi(x)$, we use partial summation:

$$\pi(x) = \int_{2^-}^{x} \frac{1}{\log t}\, d\theta(t)$$

$$= \frac{\theta(x)}{\log x} + \int_{2}^{x} \frac{\theta(t)}{t \log^2 t}\, dt.$$

$$\Rightarrow \frac{\pi(x) - li(x)}{\sqrt{x}/\log x} = \frac{\theta(x) - x}{\sqrt{x}} + o(1)$$

$$= \frac{\psi(x) - x}{\sqrt{x}} - 1 + o(1).$$

$$\begin{tikzcd} \psi \arrow[r] \arrow[d] & \theta \arrow[d] \\ \Pi \arrow[r] & \pi \end{tikzcd}$$

partial summation

$$\Pi(x) = \sum_{p^r \le x} \frac{1}{r}$$

$$= \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \dots$$

throw away prime powers (squares in particular)

For primes in arithmetic progressions,

$$\psi(x;q,a) = \sum_{\substack{p^r \le x \\ p \equiv a \ (\mathrm{mod}\ q)}} \log p$$

$$= \sum_{\substack{p \le x \\ p \equiv a \ (\mathrm{mod}\ q)}} \log p + \sum_{\substack{p^2 \le x \\ p^2 \equiv a \ (\mathrm{mod}\ q)}} \log p + \dots$$

$$= \theta(x;q,a) + \sum_{\substack{b \ (\mathrm{mod}\ q) \\ b^2 \equiv a \ (\mathrm{mod}\ q)}} \theta(x^{\frac{1}{2}};q,b) + \dots$$

If we let $c_q(a) = \#\{b \ (\mathrm{mod}\ q) : b^2 \equiv a \ (\mathrm{mod}\ q)\}$ then

$$\psi(x;q,a) = \theta(x;q,a) + c_q(a)\frac{x^{\frac{1}{2}}}{\phi(q)} + \text{error}.$$

$$\frac{\theta(x;q,a) - x/\phi(q)}{\sqrt{x}} = \frac{\psi(x;q,a) - x/\phi(q)}{\sqrt{x}} - \frac{c_q(a)}{\phi(q)} + o(1).$$