Wednesday, January 11

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} \qquad (\sigma>1)$$

$$\log \zeta(s) = \sum_p \log(1-p^{-s})^{-1}$$

$$= \sum_p \sum_{k=1}^{\infty} \frac{1}{k}(p^{-s})^k$$

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{k=1}^{\infty} \frac{1}{k}(p^{-ks})(-k\log p)$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{k=1}^{\infty} \frac{\log p}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Inverse Mellin transform

$$\psi(x) = \sum_{n\le x} \Lambda(n) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds$$

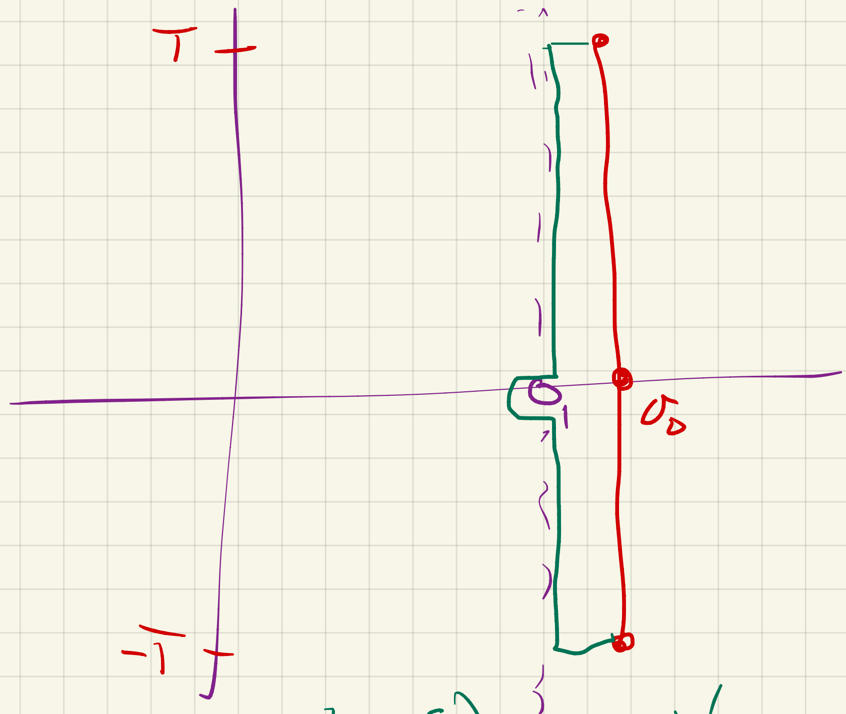
where $\sigma_0 > 1$, and the contour integral is

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds + \mathrm{Error}(\sigma_0, x, T)$$

Proving the prime number theorem

$\psi(x) \sim x$:

Variant 1: small



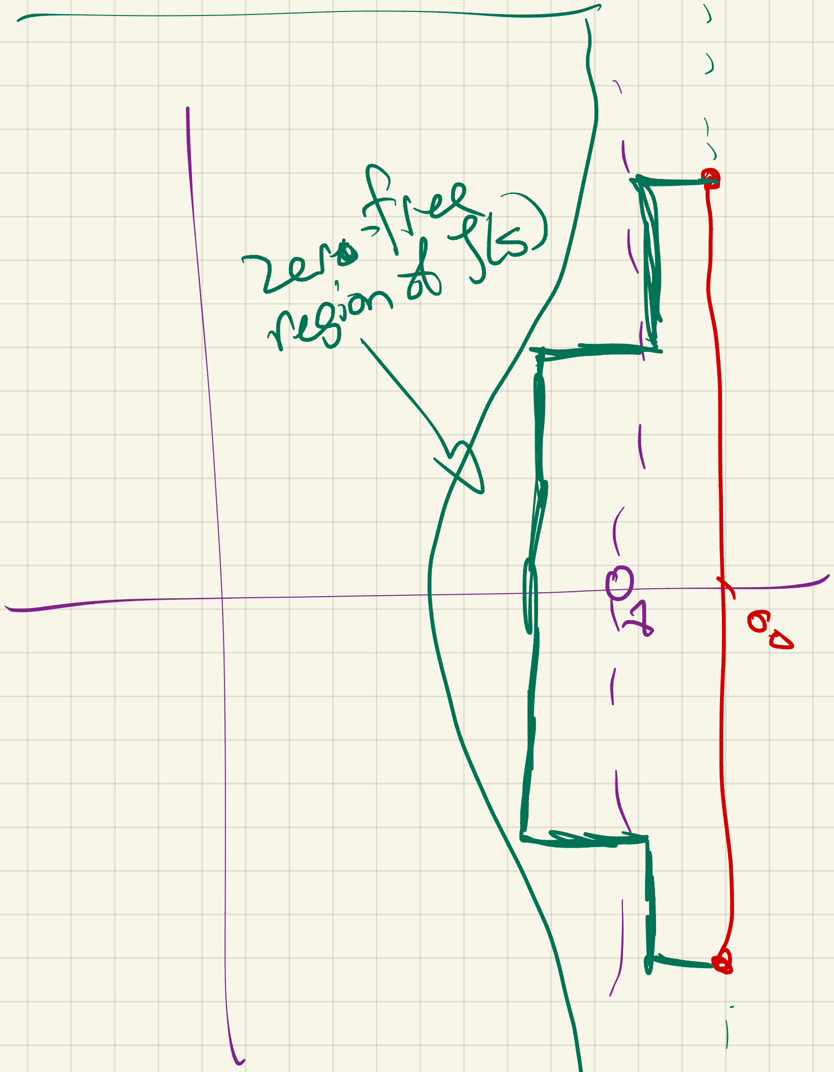
$$\mathrm{Res}_{s=1}\left(-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\right) = 1\cdot\frac{x^1}{1} = x$$

$\to$ estimate the new contour integral

$$\left[\zeta(s)\neq 0 \text{ for } \sigma=1\right]$$

$\Longleftrightarrow$ PNT

Variant 2: medium



We still get

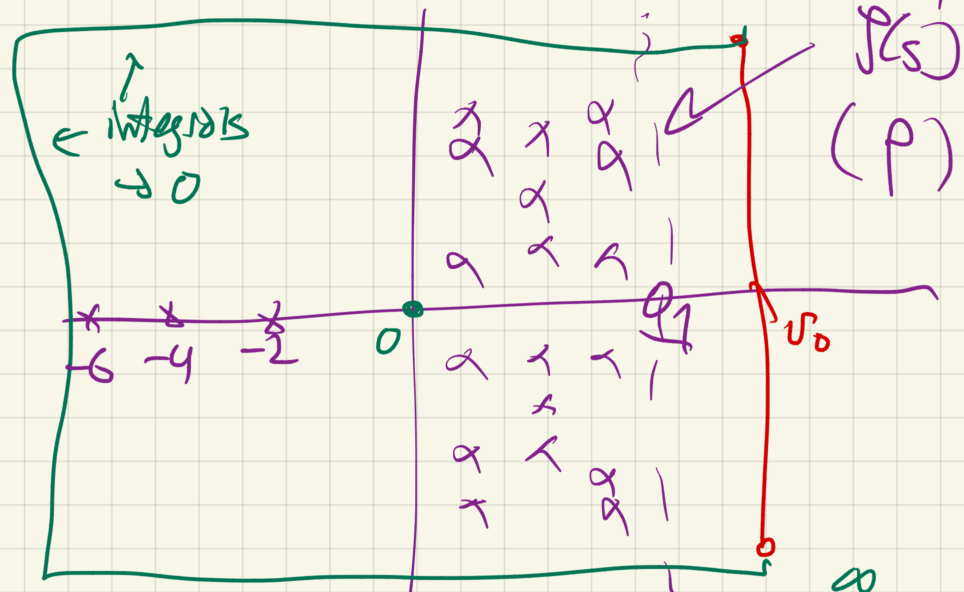
$\psi(x) = x$ + new contour integral

(+ previous error)

Estimating integral $\Rightarrow$

error is $\ll x \exp(-c\sqrt{\log x})$

Variant 3: large



$$\psi_0(x) = x - \frac{\zeta'}{\zeta}(0) - \sum_{\rho} \frac{x^\rho}{\rho} - \sum_{k=1}^{\infty} \frac{x^{-2k}}{-2k}$$

$$= x - \sum_{\substack{\rho \\ \text{nontrivial}}} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-x^{-2})$$

EQUALITY !!

"Explicit formula" for $\psi(x)$

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-x^{-2})$$

If $\rho = \beta + i\gamma$ then $x^\rho = x^\beta x^{i\gamma}$

$$= x^\beta e^{i\gamma \log x}$$

$$= x^\beta (\cos(\gamma \log x) + i \sin(\gamma \log x))$$

$$\frac{x^{\beta+i\gamma}}{\beta+i\gamma} + \frac{x^{\beta-i\gamma}}{\beta-i\gamma} = x^\beta\left(\frac{x^{i\gamma}}{\beta+i\gamma} + \frac{x^{-i\gamma}}{\beta-i\gamma}\right)$$

$$= x^\beta\left(\frac{\gamma \sin(\gamma\log x) + \cos(\gamma \log x)}{\beta^2+\gamma^2}\right)$$

$$\approx x^\beta \frac{\sin(\gamma \log x)}{\gamma}$$

On RH,

$$\psi(x) = x + O(x^{\frac{1}{2}}\log^2 x)$$

Our next topic: Dirichlet characters

- one reference: Montgomery & Vaughan Section 4.2

Start by talking about characters of finite abelian groups

- homomorphisms from G to ~~$\mathbb{C}^*$~~ S^1.

Let $\hat{G} = \{\chi : G \to S^1 : \chi$ is a group homomorphism$\}$

Ultimately we care about

$$G = (\mathbb{Z}/q\mathbb{Z})^\times \qquad q \in \mathbb{N}$$

$(\#G = \phi(q))$

$$N(T) = \#\{\rho : \zeta(\rho) = 0, \; 0 < \beta < 1, \; 0 \le \gamma \le T\}$$

$$= \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$

– Riemann – von Mangoldt formula for the zero-counting function $N(T)$ for $\zeta(s)$