Monday, January 16

- I sent out an email this weekend — email me if you didn't receive it.
- Reminder: f is multiplicative if $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$.

  f is totally multiplicative if $f(mn) = f(m)f(n)$ always.

Definition: A Dirichlet character (mod q) is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying:

(1) χ is periodic with period q;

(2) $\chi(n) \neq 0$ if and only if $(n,q) = 1$;

(3) χ is totally multiplicative.

It's easy to see that such χ are in 1-to-1 correspondence with elements of $(\mathbb{Z}/q\mathbb{Z})^\times$.

There are $\phi(q)$ Dirichlet characters (mod q), and we have the orthogonality relations:

- for any Dirichlet character χ (mod q),

$$\sum_{1 \le n \le q} \chi(n) = \begin{cases} \phi(q), & \text{if } \chi = \chi_0; \\ 0, & \text{if } \chi \neq \chi_0. \end{cases}$$

- for any $n \in \mathbb{Z}$,

$$\sum_{\chi \,(\mathrm{mod}\, q)} \chi(n) = \begin{cases} \phi(q), & \text{if } n \equiv 1 \pmod{q}, \\ 0, & \text{if } n \not\equiv 1 \pmod{q}. \end{cases}$$

All Dirichlet characters modulo 1, 2, 3, 4, 5, 12:

- $q=1$: $\chi_0 \equiv 1$
- $q=2$: $\chi_0(\text{odd})=1$, $\chi_0(\text{even})=0$
- $q=3$:

n	1	2	3
$\chi_0(n)$	1	1	0

n	1	2	3
$\chi_1(n)$	1	-1	0

- $q=4$:

n	1	2	3	4
χ_0	1	0	1	0

n	1	2	3	4
$\chi_2(n)$	1	0	-1	0

- $q=5$:

n	1	2	3	4 $\equiv -1$	5
$\chi_0(n)$	1	1	1	1	0
$\chi_3(n)$	1	i	$-i$	-1	0
$\chi_4(n)$	1	-1	-1	1	0
$\chi_5(n)$	1	$-i$	i	-1	0

- $q=12$:

n	1	2	3	4	5	6	7	8	9	10	11 $\equiv -1$	12
$\chi_0(n)$	1	0	0	0	1	0	1	0	0	0	1	0
$\chi_0 \pmod 1$	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_6(n)$	1	0	0	0	1	0	-1	0	0	0	-1	0
$\chi_2 \pmod 4$	1	0	-1	0	1	0	-1	0	1	0	-1	0
$\chi_7(n)$	1	0	0	0	-1	0	1	0	0	0	-1	0
$\chi_1 \pmod 3$	1	-1	0	1	-1	0	1	-1	0	1	-1	0
$\chi_8(n)$	1	0	0	0	-1	0	-1	0	0	0	1	0

primitive character

Note: $\chi(-n) = \chi(-1\cdot n) = \chi(-1)\chi(n)$

So: χ is an odd function if $\chi(-1)=-1$

χ is an even function if $\chi(-1)=1$.

Note: if we fix a with $(a,q)=1$:

$$\sum_{\chi \pmod q} \overline{\chi(a)}\,\chi(n) = \sum_{\chi \pmod q} \chi(a^{-1})\,\chi(n)$$

$$= \sum_{\chi \pmod q} \chi(a^{-1}n) = \begin{cases} \phi(q), & \text{if } a^{-1}n \equiv 1 \pmod q, \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \phi(q), & \text{if } n \equiv a \pmod q, \\ 0, & \text{if } n \not\equiv a \pmod q. \end{cases}$$

- looks more like "orthogonality":
- {Dirichlet characters (mod q)} is a basis for $\{f: \mathbb{Z} \to \mathbb{C} ;\ f(n)=0$ when $(n,q)>1$, $f(n+q)=f(n)\}$.

A Dirichlet character $\chi \pmod q$ is imprimitive if there exists $d \mid q$, $d<q$ such that $\chi(m)=\chi(n)$ whenever $m \equiv n \pmod d$ and $(mn,q)=1$.

- Remark: if $d \mid q$ and χ_1 is a Dirichlet character (mod d), then

$$\chi(n) = \begin{cases} \chi_1(n), & \text{if } (n,q)=1, \\ 0, & \text{if } (n,q)\neq 1 \end{cases}$$

is a Dirichlet character (mod q).

We say $\chi_1 \pmod d$ induces $\chi \pmod q$. If $d<q$, this means $\chi \pmod q$ is imprimitive.

- Indeed, $\chi \pmod q = \chi_1 \pmod d \cdot \chi_0 \pmod q$

One can show (see the discussion in MV, Section 9.1) that every $\chi \pmod q$ is induced by exactly one primitive character $\chi^* \pmod{q^*}$ for some divisor q^* of q (possible that $q^* = q$, if χ is already primitive).

$$\Rightarrow \quad \phi(q) = \sum_{d \mid q} \phi^*(d)$$

— q^* is the conductor of χ^* and χ

Theorem 4.7 (MV): Let $\chi : \mathbb{Z} \to \mathbb{C}$ have period q, and suppose χ is multiplicative and supported on n with $(n,q)=1$. Then χ is totally multiplicative, hence is a Dirichlet character.

Proof: If $(m,q)>1$ or $(n,q)>1$, then $\chi(mn) = 0 = \chi(m)\chi(n)$.

So suppose $(mn, q) = 1$.

- Choose $k \equiv \bar{q}^{-1}(1-n) \pmod m$ and set $l = n + kq$. Then $l \equiv n + \bar{q}^{-1}(1-n)q \equiv n+1-n \equiv 1 \pmod m$, and so $(l,m)=1$.

Then
$$\begin{aligned}\chi(mn) &= \chi(ml) && \text{by periodicity}\\ &= \chi(m)\chi(l) && \text{by mult'y}\\ &= \chi(m)\chi(n) && \text{by periodicity.}\end{aligned}$$

n	1	2	3	4	5	6	7	8
$\chi \pmod 8$	1	0	-1	0	1	0	-1	0
$\chi \pmod 4$	1	0	-1	0	1	0	-1	0