Monday, January 23

Warmup: recall the Maclaurin series

$$\log(1-z)^{-1} = \sum_{k=1}^{\infty} \frac{z^k}{k},$$ which "obviously" converges for $|z|<1$.

Claim: this series converges for $|z|=1$ if $z \neq 1$.

- Dirichlet's test: if $\{b_n\}$ is decreasing to 0, and $\left\{\sum_{n \le N} a_n\right\}$ is uniformly bounded, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

– Apply with $b_k = \frac{1}{k}$ and $a_k = z^k$:

when $z \neq 1$,

$$\sum_{1 \le k \le N} z^k = \frac{z - z^{N+1}}{1-z}$$ and thus

$$\left|\sum_{1 \le k \le N} z^k\right| \le \frac{2}{|1-z|}. \quad //$$

Dirichlet L-functions

For a Dirichlet character $\chi \pmod q$, define $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

(recall $s = \sigma + it \in \mathbb{C}$).

We saw on Wednesday that this series converges absolutely if and only if $\sigma > 1$.

It follows that $L(s,\chi)$ has an Euler product:

for $\sigma > 1$,

$$L(s,\chi) = \prod_p \left(1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \frac{\chi(p^3)}{p^{3s}} + \cdots\right)$$

$$= \prod_p \left(1 + \frac{\chi(p)}{p^s} + \left(\frac{\chi(p)}{p^s}\right)^2 + \left(\frac{\chi(p)}{p^s}\right)^3 + \cdots\right)$$

$$= \prod_p (1 - \chi(p)p^{-s})^{-1}. \quad (*)$$

- converges absolutely for $\sigma > 1$; therefore $L(s,\chi) \neq 0$ for $\sigma > 1$.

Example: If $\chi = \chi_0$ is principal $(\bmod q)$, so that $\chi_0(n) = \begin{cases} 1, & \text{if } (n,q) = 1, \\ 0, & \text{if } (n,q) > 1, \end{cases}$

then

$$L(s,\chi_0) = \sum_{\substack{n \geq 1 \\ (n,q)=1}} n^{-s} = \prod_p (1 - \chi_0(p)p^{-s})^{-1} =$$

$$= \prod_{p \nmid q} (1 - p^{-s})^{-1}$$

$$= \zeta(s) \prod_{p \mid q} (1 - p^{-s})$$

Conclude: $L(s,\chi_0)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$; also has a simple pole at $s=1$ with residue

$$\operatorname{Res}_{s=1}(\zeta(s)) \cdot \prod_{p \mid q}(1 - p^{-1}) = 1 \cdot \frac{\phi(q)}{q}.$$

(In fact, $\zeta(s)$ is the special case $L(s,\chi_0)$ with $\chi_0 \pmod 1$)

For $\sigma > 1$, take logarithms of $(*)$:

$$\log L(s,\chi) = \sum_p \log(1 - \chi(p)p^{-s})^{-1}$$

$$= \sum_p \sum_{k=1}^{\infty} \frac{(\chi(p)p^{-s})^k}{k} =$$

$$\log L(s,\chi) = \sum_p \log\left(1-\chi(p)p^{-s}\right)^{-1}$$

$$= \sum_p \sum_{k=1}^{\infty} \frac{(\chi(p)p^{-s})^k}{k} =$$

$$= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\chi(n)n^{-s}, \quad (\sigma>1)$$

where $\Lambda(n) = \begin{cases} \log p, & \text{if } n=p^k, \\ 0, & \text{otherwise.} \end{cases}$

Hence, taking derivatives,

$$\frac{L'}{L}(s,\chi) = -\sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s}. \quad (\sigma>1)$$

Theorem 4.8 (MV): If $\chi \neq \chi_0$, then $\sum_{n=1}^{\infty} \chi(n)n^{-s}$ converges for $\sigma>0$.

- Consequence: $L(1,\chi) \neq \infty$ for $\chi \neq \chi_0$.

Proof 1 (Leo): define $S_\chi(t) = \sum_{n\le t} \chi(n)$,

then by partial summation,

$$\sum_{n=1}^{\infty} \chi(n)n^{-s} = s\int_1^{\infty} S_\chi(t)\,t^{-s-1}\,dt;$$

since $S_\chi(t)$ is bounded, this integral converges (absolutely) for $\sigma>0$.

Proof 2: By Theorem 1.3 (MV), the abscissa of convergence, σ_c, satisfies

$$\sigma_c = \limsup_{x\to\infty} \frac{\log|S_\chi(x)|}{\log x} = 0.$$

Recall the Gauss sum

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a)\,e(a/q).$$

We proved:

- If χ is primitive (mod q), then
  $|\tau(\chi)| = \sqrt{q}$.
- Theorem 9.5 (MV). If χ is primitive
  or if $(n,q)=1$, then

$$\boxed{\chi(n)\,\tau(\bar{\chi}) = \sum_{a=1}^{q} \bar{\chi}(a)\, e\left(\frac{an}{q}\right)}$$

In particular, if $n=-1$:

$$\chi(-1)\,\tau(\bar{\chi}) = \sum_{a=1}^{q} \bar{\chi}(a)\, e\left(\frac{-a}{q}\right)$$

$$= \sum_{a=1}^{q} \chi(a)\, e\left(\frac{a}{q}\right) = \overline{\tau(\chi)}.$$

Multiplying by $\tau(\chi)$:

$$\chi(-1)\,\tau(\bar{\chi})\,\tau(\chi) = \overline{\tau(\chi)}\,\tau(\chi) = |\tau(\chi)|^2 = q.$$

(for any χ)

Let's examine $L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$

for $\chi \pmod q$ primitive, $q>1$.

By Theorem 9.5,

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(a)\, e\left(\frac{an}{q}\right)$$

$$= \frac{\chi(-1)\,\tau(\chi)}{q} \sum_{a=1}^{q} \bar{\chi}(a)\, e\left(\frac{an}{q}\right);$$

and so

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{\chi(-1)\,\tau(\chi)}{q} \sum_{a=1}^{q} \bar{\chi}(a)\, e\left(\frac{an}{q}\right)$$

$$= \frac{\chi(-1)\,\tau(\chi)}{q} \sum_{a=1}^{\cancel{q}\ q-1} \bar{\chi}(a) \sum_{n=1}^{\infty} \frac{1}{n} e\left(\frac{an}{q}\right)$$

$$= \frac{\chi(-1)\,\tau(\chi)}{q} \sum_{a=1}^{q-1} \bar{\chi}(a) \log\left(1-e\left(\frac{a}{q}\right)\right)^{-1}.$$

- finite sum!

$\cancel{L(s,\chi)} =$

$$L(1,\chi) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{a=1}^{q-1} \overline{\chi}(a) \log\left(1 - e\left(\frac{a}{q}\right)\right)^{-1}.$$

Using
$$1 - e(\theta) = -2i\, e\left(\frac{\theta}{2}\right)\left(\frac{e\left(\frac{\theta}{2}\right) - e\left(-\frac{\theta}{2}\right)}{2i}\right)$$
$$= -2i\, e(\theta/2)\sin(\pi\theta),$$

we have

$$\cancel{L(s,\chi)}\ L(1,\chi) = \frac{\chi(-1)\tau(\chi)}{q} \sum_{a=1}^{q-1} \overline{\chi}(a) \log\left(-2i\, e\left(\frac{a}{2q}\right)\sin\left(\pi a/q\right)\right)^{-1}$$

$$= -\frac{\chi(-1)\tau(\chi)}{q} \sum_{a=1}^{q-1} \overline{\chi}(a) \cdot \left(\underbrace{\cancel{\log(-2i)}}_{0} + \pi i \frac{a}{q} + \log\left(\sin\frac{\pi a}{q}\right)\right)$$

by orthogonality

$$\left[\log e(\theta) = \log\left(e^{2\pi i \theta}\right) = 2\pi i \theta\right]$$

Conclusion:

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{q} \cdot \left(\sum_{a=1}^{q-1} \overline{\chi}(a)\, \pi i \frac{a}{q} + \sum_{a=1}^{q-1} \overline{\chi}(a) \log\left(\sin\frac{\pi a}{q}\right)\right).$$

$$\sum_{a=1}^{q-1} \overline{\chi}(a) \log(-2i) = \log(-2i) \underbrace{\sum_{a=1}^{q} \overline{\chi}(a)}_{=0.}$$