# Wednesday, January 25

On Monday, we had shown that if $\chi \pmod q$ is primitive and $q>1$, then

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{q}(S+iT), \text{ where}$$

$$S = \sum_{a=1}^{q-1} \bar{\chi}(a)\log\left(\sin\frac{\pi a}{q}\right)$$

$$T = \frac{\pi}{q}\sum_{a=1}^{q-1}\bar{\chi}(a)\, a.$$

Note: since $q-a$ runs over $\{1,2,\dots,q-1\}$ as $a$ does,

$$S = \sum_{a=1}^{q-1}\bar{\chi}(q-a)\log\left(\sin\frac{\pi(q-a)}{q}\right)$$

$$= \bar{\chi}(-1)\sum_{a=1}^{q-1}\bar{\chi}(a)\log\left(\sin\frac{\pi a}{q}\right) = \chi(-1)S.$$

In particular, if $\chi$ is odd then $S=0$.

$$T = \frac{\pi}{q}\sum_{a=1}^{q-1}\bar{\chi}(q-a)(q-a)$$

$$= \chi(-1)\frac{\pi}{q}\left(\cancel{\sum_{a=1}^{q-1}\bar{\chi}(a)\,q}_{\text{0 by orthogonality}} - \sum_{a=1}^{q-1}\bar{\chi}(a)\,a\right)$$

$$= -\chi(-1)T.$$

In particular, if $\chi$ is even, then $T=0$.

<u>Theorem 9.9 (MV)</u>: Let $\chi \pmod q$ be primitive, $q>1$. If $\chi(-1)=1$,

$$L(1,\chi) = -\frac{\tau(\chi)}{q}\sum_{a=1}^{q-1}\bar{\chi}(a)\log\left(\sin\frac{\pi a}{q}\right);$$

if $\chi(-1)=-1$,

$$L(1,\chi) = \frac{i\pi\tau(\chi)}{q^2}\sum_{a=1}^{q-1}\bar{\chi}(a)\,a.$$

If $\chi(-1)=1$,

$$L(1,\chi) = -\frac{\tau(\chi)}{q}\sum_{a=1}^{q-1}\bar{\chi}(a)\log\left|\sin\frac{\pi a}{q}\right|;$$

if $\chi(-1)=-1$,

$$L(1,\chi) = \frac{i\pi\tau(\chi)}{q^2}\sum_{a=1}^{q-1}\bar{\chi}(a)a.$$

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Examples!

- $q=4$, $\chi(1)=1$, $\chi(3)=-1$

$$\tau(\chi) = \sum_{a=1}^{4}\chi(a)e\left(\frac{a}{4}\right) = 1e^{2\pi i/4} - 1\cdot e^{2\pi i\cdot 3/4}$$

$$= 1\cdot i - 1\cdot(-i) = 2i.$$

So

$$L(1,\chi) = \frac{i\pi\cdot 2i}{4^2}(1\cdot 1 - 1\cdot 3) = \frac{\pi}{4}.$$

Note: $L(1,\chi) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$

- $q=3$, $\chi(1)=1$, $\chi(2)=-1$

$$\tau(\chi) = \sum_{a=1}^{3}\chi(a)e\left(\frac{a}{3}\right)$$

$$= 1e^{2\pi i/3} - 1e^{2\pi i\cdot 2/3}$$

$$= \frac{-1+i\sqrt{3}}{2} - \left(\frac{-1-i\sqrt{3}}{2}\right) = i\sqrt{3}.$$

So

$$L(1,\chi) = \frac{i\pi\cdot i\sqrt{3}}{3^2}(1\cdot 1 - 1\cdot 2)$$

$$= \pi/3\sqrt{3}.$$

$$L(1,\chi) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \cdots$$

- $\chi \pmod 5$, $\chi(1)=1$, $\chi(2)=-1$, $\chi(3)=-1$, $\chi(4)=1$

$$L(1,\chi) = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \cdots$$

$$= \frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}}{2}.$$

- $q=7$, $\chi(1)=\chi(2)=\chi(4)=1$, $\chi(3)=\chi(5)=\chi(6)=-1$.

$$L(1,\chi)=\frac{i\pi\,\tau(\chi)}{7^2}(1+2-3+4-5-6)$$

$$=\cdots=\frac{\pi}{\sqrt{7}}.$$

Side note: replace $\frac{1}{n}$ by $\int_0^1 t^{n-1}\,dt$:

$$L(1,\chi)=\sum_{n=1}^{\infty}\chi(n)\int_0^1 t^{n-1}\,dt$$

$$=\sum_{a=1}^{q}\chi(a)\int_0^1\sum_{k=0}^{\infty}t^{(kq+a)-1}\,dt$$

$$=\sum_{a=1}^{q}\chi(a)\int_0^1\frac{t^{a-1}}{1-t^q}\,dt$$

$$=\int_0^1\frac{\sum_{a=1}^{q}\chi(a)t^{a-1}}{1-t^q}\,dt. \quad \leftarrow \star$$

$\star$ polynomial, divisible by $1-t$ when $\chi\neq\chi_0$. So can cancel $1-t$ from denominator.

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## Theorem 4.9 (MV) – Dirichlet

If $\chi$ is nonprincipal then $L(1,\chi)\neq 0$.

Assume this theorem for the moment.

Recall: if $(a,q)=1$,

$$\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\overline{\chi}(a)\chi(n)=\begin{cases}1, & \text{if } n\equiv a \pmod q,\\ 0, & \text{if } n\not\equiv a \pmod q\end{cases}$$

So

(for $\sigma > 1$)

$$\sum_{\substack{n\geq 1 \\ n\equiv a \,(\mathrm{mod}\, q)}} \frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\, q)} \overline{\chi}(a)\chi(n)$$

$$= \frac{1}{\phi(q)} \sum_{\chi(\mathrm{mod}\, q)} \overline{\chi}(a) \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \chi(n)$$

$$= \frac{1}{\phi(q)} \left( -\frac{L'}{L}(s,\chi_0) - \sum_{\chi\neq\chi_0} \overline{\chi}(a) \frac{L'}{L}(s,\chi) \right).$$

Consider what happens for $s\in\mathbb{R}$, $s \searrow 1$.
$s\to 1^+$.

Since $L(s,\chi_0)$ has a simple pole at $s=1$,

$$-\frac{L'}{L}(s,\chi_0) = \frac{1}{s-1} + O_q(1) \quad \text{near } s=1.$$

Since the other $L(s,\chi)$ are nonzero at $s=1$,

$$\frac{L'}{L}(s,\chi) \ll_q 1 \quad \text{near } s=1.$$

So as $s\to 1^+$:

$$\sum_{\substack{n\geq 1 \\ n\equiv a \,(\mathrm{mod}\, q)}} \frac{\Lambda(n)}{n^s} = \frac{1}{\phi(q)} \frac{1}{s-1} + O_q(1).$$

$\to \infty$ as $s\to 1^+$.

Consequently,

$$\sum_{\substack{n\geq 1 \\ n\equiv a \,(\mathrm{mod}\, q)}} \frac{\Lambda(n)}{n} = \infty.$$

For the proper prime powers,

$$\sum_{k=2}^{\infty} \sum_{p^k\equiv a \,(\mathrm{mod}\, q)} \frac{\Lambda(p^k)}{p^k} \leq \sum_{k\geq 2} \sum_{p} \frac{\log p}{p^k}$$

$$= \sum_p \log p \sum_{k\geq 2} \frac{1}{p^k} = \sum_p \frac{\log p}{p(p-1)} < \infty.$$

Hence $$\sum_{p\equiv a \,(\mathrm{mod}\, q)} \frac{\log p}{p} = \infty.$$

Summary: assuming $L(1,\chi)\neq 0$:

Corollary 4.10 (MV) - Dirichlet

If $(a,q)=1$, then $\sum_{p\equiv a \pmod q} \frac{\log p}{p}$ diverges. Hence there are infinitely many primes congruent to $a \pmod q$.

Let's turn to proving that $L(1,\chi)\neq 0$.

It turns out that complex characters are easier than real (quadratic) characters.

Proof for complex $\chi$: for $\sigma>1$,

$$\prod_{\chi \pmod q} L(s,\chi) = \exp\left(\sum_{\chi} \log L(s,\chi)\right)$$

$$= \exp\left(\sum_{\chi}\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n}\chi(n) n^{-s}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \sum_{\chi \pmod q} \chi(n)\right)$$

$$= \exp\left(\underbrace{\phi(q)\sum_{n\equiv 1 \pmod q} \frac{\Lambda(n)}{\log n} n^{-s}}_{\text{if } s\in\mathbb{R},\ s>1:\ \text{nonnegative.}}\right)$$

Hence $\prod_{\chi \pmod q} L(s,\chi) \geq 1$ when $s>1$.

It's impossible for two $L(s,\chi)$'s to have zeros at $s=1$, since then $\prod_{\chi \pmod q} L(s,\chi)$ would vanish at $s=1$. (but continuous $s\to 1^+$)

But also: if $L(1,\chi)=0$, then
$L(1,\bar{\chi})=0$.

Hence: if $\chi$ is complex, then
$L(1,\chi)\neq 0$.

[On Friday, we'll prove that
$L(1,\chi)\neq 0$ for quadratic (real) $\chi$.]

- Side note: the function
$$\prod_{\chi \,(\mathrm{mod}\, q)} L(s,\chi)$$
happens to be a "Dedekind zeta function", for the number field $\mathbb{Q}(e^{2\pi i/q})$.