Friday, January 27

Vivanti-Pringsheim Theorem:

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$ be a power series centred at $c \in \mathbb{R}$, with (finite) radius of convergence $R > 0$. Suppose that $a_n \geq 0$ for all n. Then $f(z)$ has a singularity at $z = c + R$.

("singularity" means there's no analytic continuation to a neighbourhood of $c+R$).

Landau's Theorem (Theorem 1.7, MV)

Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ have (finite) abscissa of convergence σ_c.

If $a_n \geq 0$ for all n, then $\alpha(s)$ has a singularity at $s = \sigma_c$.

Equivalently: if $\alpha(s)$ has an analytic continuation to $\{\mathrm{Re}(s) > \sigma_0\}$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ converges for $\sigma > \sigma_0$.

[Consider $\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}$
$= (1 - 2^{1-s}) \zeta(s)$]

Recall, on Wednesday:

- we showed that "$L(1,\chi)$ is never 0" implies Dirichlet's theorem on primes in arithmetic progressions
- we proved that $L(1,\chi) \neq 0$ for complex characters χ

Now let's prove $L(1,\chi) \neq 0$ for quadratic characters χ.

<u>Lemma</u>: Let $r = \chi * 1$; that is,

$$r(n) = \sum_{d \mid n} \chi(d)\, 1\left(\tfrac{n}{d}\right) = \sum_{d \mid n} \chi(d).$$

Then $r(n) \geq 0$, and $r(n) \geq 1$ for all $n = m^2$.

<u>Proof</u>: Since $\chi, 1$ are multiplicative, so is r; so it suffices to show this when $n = p^k$.

	1	p	p^2	p^3	p^4	p^5	...
1	1	1	1	1	1	1	
if $\chi(p)=1$:	1	1	1	1	1	1	...
$r=\chi*1$	1	2	3	4	5	6	...
if $\chi(p)=-1$:	1	-1	1	-1	1	-1	
$r=\chi*1$:	1	0	1	0	1	0	

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Let's use this to prove $L(1,\chi) \neq 0$!

Let $f(s) = \zeta(s) L(s,\chi)$, so

that $f(s) = \sum_{n=1}^{\infty} (1 * \chi) n^{-s} = \sum_{n=1}^{\infty} r(n) n^{-s}$.

If $L(1,\chi) = 0$, then $f(s)$

is analytic for $\sigma > 0$.

Since $r(n) \geq 0$, by Landau's theorem,

$\sum_{n=1}^{\infty} r(n) n^{-s}$ converges for $\sigma > 0$,

But, at $s = \frac{1}{2}$:

$$\sum_{n=1}^{\infty} r(n) n^{-\frac{1}{2}} \geq \sum_{m=1}^{\infty} r(m^2)(m^2)^{-\frac{1}{2}}$$

$$\geq \sum_{m=1}^{\infty} 1 \, m^{-1} = \infty,$$

a contradiction.

[Wed: we saw

$$\sum_{p \equiv a \pmod q} \frac{\log p}{p} = \infty.$$

Mertens-like results:

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_q(1).$$

Indeed

$$\sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \frac{1}{p} = \frac{1}{\phi(q)} \log\log x + b(a,q) + O_q\left(\frac{1}{\log x}\right),$$]

Poisson summation formula:

Let $f \in L^1(\mathbb{R})$, and define its Fourier transform

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) e(-tx)\, dx$$

If $f(x)$ is of bounded variation on $\mathbb{R}$, and $f(x)$ is continuous, then

$$\sum_{n \in \mathbb{Z}} f(n) = \lim_{M \to \infty} \sum_{m=-M}^{M} \hat{f}(m).$$

One application: if

$$f(x) = e^{-\pi x^2 z},$$

then one can show

$$\hat{f}(t) = z^{-1/2} e^{-\pi t^2 / z}.$$

Theorem 10.1 (MV): For $\alpha \in \mathbb{R}$ and z with $\mathrm{Re}\, z > 0$:

$$\sum_{n=-\infty}^{\infty} e^{-\pi (n+\alpha)^2 z} = z^{-1/2} \sum_{k=-\infty}^{\infty} e(k\alpha) e^{-\pi k^2 / z}$$

and

$$\sum_{n=-\infty}^{\infty} (n+\alpha) e^{-\pi (n+\alpha)^2 z} = -i z^{-3/2} \sum_{k=-\infty}^{\infty} k\, e(k\alpha) e^{-\pi k^2 / z}.$$

(The branch of $z^{1/2}$ is determined by $1^{1/2} = 1$.)

Definition: For $\mathrm{Re}\, z>0$ and $\chi \pmod q$, define

$$\theta_0(z,\chi) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 z/q}$$

$$\theta_1(z,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n) e^{-\pi n^2 z/q}.$$

Remark: If $\chi(-1)=1$, then $\theta_1(z,\chi)\equiv 0$ identically; if $\chi(-1)=-1$, then $\theta_0(z,\chi)\equiv 0$.

Also, $$|\theta_0(z,\chi)| \le \sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty} e^{-\pi n^2 (\mathrm{Re}\, z)/q}$$

$$\le 2\sum_{n=1}^{\infty} e^{-\pi n (\mathrm{Re}\, z)/q} \ll_{q,\delta} e^{-\pi(\mathrm{Re}\, z)/q}$$

uniformly for $\mathrm{Re}\, z \ge \delta$.

Theorem 10.8 (MV) For $\mathrm{Re}\, z>0$ and $\chi \pmod q$ primitive,

$$\theta_0(z,\chi) = \frac{\tau(\chi)}{\sqrt{q}} z^{-1/2}\, \theta_0\left(\frac{1}{z},\overline{\chi}\right)$$

$$\theta_1(z,\chi) = \frac{\tau(\chi)}{i\sqrt{q}} z^{-3/2}\, \theta_1\left(\frac{1}{z},\overline{\chi}\right).$$

Sketch:

Proof (for θ_0): since χ has period q,

$$\theta_0(z,\chi) = \sum_{a \pmod q} \chi(a) \sum_{\substack{n\in\mathbb{Z} \\ n\equiv a \pmod q}} e^{-\pi n^2 z/q}$$

$$= \sum_{a \pmod q} \chi(a) \sum_{m\in\mathbb{Z}} e^{-\pi (mq+a)^2 z/q},$$

to which we can apply Theorem 10.1 with $\alpha = a/q$. Some factor $e(k\frac{a}{q})$ gives $\sum_{a \pmod q} \chi(a) e(k\frac{a}{q}) = \overline{\chi}(k)\tau(\chi)$.