Monday, January 30

Recall from Friday: we had defined for $\mathrm{Re}\, z > 0$ and $\chi \pmod q$:

$$\theta_0(z,\chi) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 z/q}$$

$$\theta_1(z,\chi) = \sum_{n=-\infty}^{\infty} n\chi(n) e^{-\pi n^2 z/q}.$$

- If χ is odd, then $\theta_0(z,\chi) \equiv 0$ and
$$\theta_1(z,\chi) = 2\sum_{n=1}^{\infty} n\chi(n) e^{-\pi n^2 z/q}$$
- If χ is even, then $\theta_1(z,\chi) \equiv 0$ and
$$\theta_0(z,\chi) = 2\sum_{n=1}^{\infty} \chi(n) e^{-\pi n^2 z/q} \boxed{+1 \text{ if } q=1.}$$
- Both $\theta_0(z,\chi)$ and $\theta_1(z,\chi)$ are $\ll_q e^{-\pi^2 \mathrm{Re}(z)/q}$ uniformly for $\mathrm{Re}\, z \geq 1$.

We had proved that if χ is primitive,

$$\theta_0(z,\chi) = \frac{\tau(\chi)}{\sqrt{q}} z^{-\frac{1}{2}} \theta_0\left(\tfrac{1}{z}, \bar\chi\right)$$

$$\theta_1(z,\chi) = \frac{\tau(\chi)}{i\sqrt{q}} z^{-3/2} \theta_1\left(\tfrac{1}{z}, \bar\chi\right).$$

Notation: define $\kappa = \kappa(\chi)$ by

$$\kappa = \begin{cases} 0, & \text{if } \chi \text{ is even,} \\ 1, & \text{if } \chi \text{ is odd,} \end{cases}$$

so that $\chi(-1) = (-1)^\kappa$. Define

$$\varepsilon(\chi) = \frac{\tau(\chi)}{i^\kappa \sqrt{q}};$$ note that

$$\varepsilon(\chi)\varepsilon(\bar\chi) = 1 \quad \text{and} \quad |\varepsilon(\chi)| = 1$$
when χ is primitive.

Then we can write

$$\theta_\kappa(z,\chi) = \varepsilon(\chi) z^{-\frac{1}{2}-\kappa} \theta_\kappa\left(\tfrac{1}{z}, \bar\chi\right).$$

"Recall" that $\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}$

which converges for $\operatorname{Re} s > 0$.

- $\Gamma(s)$ has a meromorphic continuation to $\mathbb{C}$, with poles at $s = 0, -1, -2, \ldots$
- $\Gamma(s) \neq 0$
- $\Gamma(1) = 1$
- $\Gamma(s+1) = s\Gamma(s)$
- $\Gamma(n) = (n-1)!$ for $n \geq 1$.

Proof of the functional equation for $L(s,\chi)$, assuming χ is primitive and $q > 1$.

(I'm following Davenport for this proof.)

Assume $\sigma > 1$:

$$\Gamma\left(\frac{s+k}{2}\right) = \int_0^\infty e^{-x} x^{\frac{s+k}{2}} \frac{dx}{x}.$$

Change variables $x = \frac{\pi n^2}{q} u$: for any $n \in \mathbb{N}$:

$$\Gamma\left(\frac{s+k}{2}\right) = \int_0^\infty e^{-\pi n^2 u/q} \left(\frac{\pi n^2 u}{q}\right)^{\frac{s+k}{2}} \frac{du}{u}$$

$$n^{-s} \Gamma\left(\frac{s+k}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{s+k}{2}} = \int_0^\infty n^k e^{-\pi n^2 u/q} \cdot u^{\frac{s+k}{2}} \frac{du}{u}.$$

Multiply by $\chi(n)$ and sum over $n \in \mathbb{N}$:

$$L(s,\chi) \Gamma\left(\frac{s+k}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{s+k}{2}} = \int_0^\infty \frac{1}{2} \theta_k(u,\chi) u^{\frac{s+k}{2}} \frac{du}{u} = \int_0^1 + \int_1^\infty.$$

We have (setting $v = \frac{1}{u}$ eventually)

$$\int_0^1 \frac{1}{2} \theta_k(u,\chi) u^{\frac{s+k}{2}} \frac{du}{u} = \int_0^1 \frac{1}{2} \varepsilon(\chi) u^{-\frac{1}{2}-k} \theta_k\left(\frac{1}{u}, \bar{\chi}\right) u^{\frac{s+k}{2}} \frac{du}{u}$$

$$= \int_1^\infty \frac{1}{2} \varepsilon(\chi) v^{\frac{1-s+k}{2}} \theta_k(v,\bar{\chi}) \frac{dv}{v}.$$

We've shown for $\sigma > 1$:

$$L(s,\chi)\,\Gamma\!\left(\frac{s+k}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{s+k}{2}}$$

$$= \int_1^\infty \frac{1}{2}\,\theta_k(u,\chi)\, u^{\frac{s+k}{2}}\,\frac{du}{u}$$

$$+\ \varepsilon(\chi)\int_1^\infty \frac{1}{2}\,\theta_k(v,\bar{\chi})\, v^{\frac{1-s+k}{2}}\,\frac{dv}{v}.$$

We observe on the RHS:

- converges for all $s \in \mathbb{C}$; gives analytic continuation of $L(s,\chi)$ to all of $\mathbb{C}$
- if we change s to $1-s$ and χ to $\bar{\chi}$, and multiply RHS by $\varepsilon(\chi)$, the RHS is unchanged!

"Corollary": For all $s \in \mathbb{C}$, $\boxed{\chi \pmod q \text{ is primitive}}$

$$L(s,\chi)\,\Gamma\!\left(\frac{s+k}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{s+k}{2}}$$

$$= \varepsilon(\chi)\,L(1-s,\bar{\chi})\,\Gamma\!\left(\frac{1-s+k}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{1-s+k}{2}}$$

Functional equation for $L(s,\chi)$.

If we define

$$\xi(s,\chi) = L(s,\chi)\,\Gamma\!\left(\frac{s+k}{2}\right)\left(\frac{q}{\pi}\right)^{\frac{s+k}{2}}$$

then $\xi(s,\chi)$ is entire and

$$\xi(s,\chi) = \varepsilon(\chi)\,\xi(1-s,\bar{\chi}).$$

Asymmetric form:

$$L(s,\chi) = \varepsilon(\chi)\,L(1-s,\bar{\chi})\,2^s\,\pi^{s-1}\,q^{\frac{1}{2}-s}\,\Gamma(1-s)\,\sin\frac{\pi}{2}(s+k).$$

* See next page →

Note: if $\chi \pmod q$ is induced by $\chi^* \pmod{q^*}$, then $L(s,\chi) = L(s,\chi^*) \prod_{\substack{p \mid q \\ p \nmid q^*}} (1 - \chi^*(p) p^{-s})$.

$\underbrace{(1 - \chi^*(p) p^{-s})}$ equals 0 when $p^s = \chi^*(p)$ (necessarily $\sigma = 0$)

vertical AP of zeros

$L(1-\bar\rho,\chi)=0$

$L(\rho,\chi)=0$

$L(\rho,\chi)=0$

$\sigma>1$: $L(s,\chi) \neq 0$ by Euler product

$\frac{1}{2}$

1

no pole ($q>1$)

$L(s,\chi)=0$ only when:

- $s = 0, -2, -4, \ldots$ if χ is even;
- $s = -1, -3, -5, \ldots$ if χ is odd.

$L(\bar\rho,\bar\chi)=0$

$L(1-\rho,\bar\chi)=0$

$L(\bar\rho,\bar\chi)=0$

* Γ factors have been simplified using:

- reflection $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$
- duplication $\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\, 2^{1-2z}\, \Gamma(2z)$