MATH 613D: Analytic Number Theory II

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Please send me your completed survey form today (if you haven't already).

Prime counting functions:

$$\pi(x) = \#\{\text{primes} \le x\} = \sum_{p \le x} 1$$

$$\theta(x) = \sum_{p \le x} \log p$$

$$\psi(x) = \sum_{n \le x} \Lambda(n), \text{ where}$$

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^r \text{ for some } r \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

- $\theta(x)$ is roughly $\pi(x) \log x$
- $\psi(x) = \theta(x) + \theta(x^{\frac{1}{2}}) + \theta(x^{\frac{1}{3}}) + \cdots$

$$= \theta(x) + \theta(x^{\frac{1}{2}}) + \text{negligible}$$

Prime number theorem:

$\psi(x) \sim x$ (meaning $\lim_{x \to \infty} \frac{\psi(x)}{x} = 1$)

$\theta(x) \sim x$

$\pi(x) \sim Li(x)$, where

$$Li(x) = \int_2^x \frac{dt}{\log t}.$$

In fact, $Li(x) \sim x/\log x$; so

$$\pi(x) \sim x/\log x.$$

Indeed: $Li(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \cdots$

Turns out:

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

$$\pi(x) = Li(x) + \underbrace{O\left(x \exp\left(-c\sqrt{\log x}\right)\right)}_{\text{for some absolute constant } c>0.}$$

$$\Downarrow$$

$$\pi(x) = Li(x) + O_A\left(x/\log^A x\right)$$

Assuming Riemann Hypothesis:

$$\psi(x) = x + O(x^{\frac{1}{2}} \log^2 x)$$

$$\theta(x) = x + O(x^{\frac{1}{2}} \log^2 x)$$

$$\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}} \log x)$$

Riemann zeta function: function of a complex variable $s = \sigma + it$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$ converges (absolutely) for $\sigma > 1$ ($\mathrm{Re}\, s > 1$).

Euler product:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

converges absolutely for $\sigma > 1$

$\Rightarrow \zeta(s) \neq 0$ for $\sigma > 1$.

$\zeta(s)$ extends by analytic continuation to $\mathbb{C} \setminus \{1\}$. $\zeta(s)$ has a simple pole at $s = 1$, of residue 1.

Functional equation: Define

$$\xi(s) = \frac{1}{2} s(s-1) \zeta(s) \Gamma(\tfrac{s}{2}) \pi^{-s/2}.$$

Then $\xi(s)$ is entire, and

$$\xi(s) = \xi(1-s).$$

$\Rightarrow \zeta(s) \neq 0$ for $\sigma < 0$.

except for $\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0$.

"trivial zeros"

Region of interest is the "critical strip"

$$\{s \in \mathbb{C} : 0 \leq \sigma \leq 1\}$$

"zero-free region"

$\zeta(s) \neq 0$ when

$$\sigma > 1 - \frac{c}{\log t}$$