Wednesday, March 1

Sources for today's lecture:

- "Inclusive": Inclusive prime number races, with Nathan Ng
- "Inequities": Inequities in the Shanks–Rényi prime number race: an asymptotic formula for the densities, with Daniel Fiorilli

Definition 2.1 (Inclusive): Let $h : [1,\infty) \to \mathbb{R}^r$ (or $\mathbb{C}^r$). The *limiting logarithmic distribution* of h is a measure on $\mathbb{R}^r$ such that, for all bounded continuous functions $f : \mathbb{R}^r \to \mathbb{R}$,

$$\lim_{x\to\infty} \frac{1}{\log x} \int_1^x f(h(t))\,\frac{dt}{t} = \int_{\mathbb{R}^r} f(\vec{y})\,d\mu(\vec{y}).$$

- By a change of variables, this is the same as the limiting distribution of $h(e^t)$:

$$\lim_{Z\to\infty} \frac{1}{Z} \int_0^Z f(h(e^t))\,dt = \int_{\mathbb{R}^r} f(\vec{y})\,d\mu(\vec{y}).$$

<u>Example:</u> If $h(x) = x^{i\gamma} = e^{i\gamma \log x}$, then μ is Haar measure on the unit circle in $\mathbb{C}$. Note

$$\lim_{x\to\infty} \frac{1}{\log x} \int_1^x f(t^{i\gamma})\,\frac{dt}{t} = \underbrace{\int f(y)\,dH(y)}$$

is the expectation of $f(Z)$, where Z is a random variable that's uniformly distributed on S^1.

Warmup: Let's compute the limiting logarithmic distribution of

$$\eta(x) = c x^{ir} + \bar{c} x^{-ir} = 2\,\mathrm{Re}(c x^{ir}) \in \mathbb{R}.$$

We'll compute its Fourier transform: by definition, the Fourier transform of a measure μ on $\mathbb{R}^n$ is

$$\hat{\mu}(\vec{t}) = \int_{\mathbb{R}^n} e^{i\vec{t}\cdot\vec{x}}\, d\mu(\vec{x}).$$

Side note: In principle, $\hat{\mu}$ contains all the information about μ.

For example:

$$\mu([3,4]) = \int_3^4 d\mu(t) = \int_{\mathbb{R}} \mathbb{1}_{[3,4]}(t)\, d\mu(t);$$

by Parseval's identity, this equals

$$\int_{\mathbb{R}} \hat{\mathbb{1}}_{[3,4]}(u)\, \hat{\mu}(u)\, du$$

(This computation assumes that μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.)

We have $\eta(x) = 2\,\mathrm{Re}(c\, h(x))$ where $h(x) = x^{ir}$; therefore

$$\lim_{X\to\infty} \frac{1}{\log X} \int_1^X 2\,\mathrm{Re}(c\,h(t))\,\frac{dt}{t} = \int_{S^1} 2\,\mathrm{Re}(cy)\, d\mathrm{Haar}(y)$$

abandon this...

Take $f(\alpha) = e^{it\cdot\eta(\alpha)} = e^{it\cdot 2\mathrm{Re}(c\alpha)}$

where $\alpha \in S^1$. Then

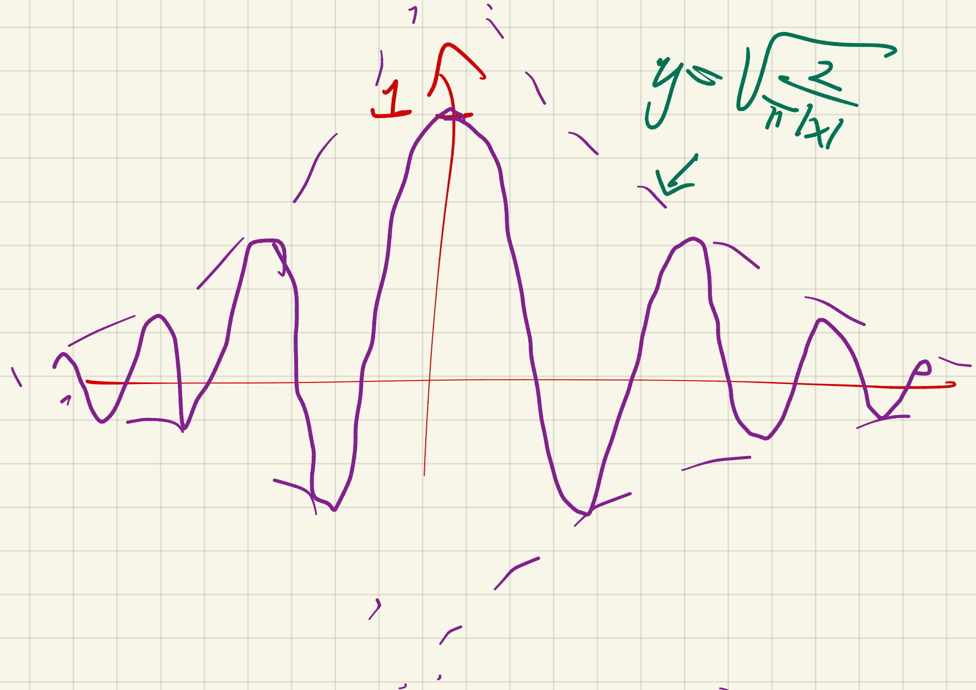
$$\hat{\eta}(t) = \int e^{it\cdot 2\mathrm{Re}(c\alpha)}\, d\mathrm{Haar}(\alpha)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{it\cdot 2\mathrm{Re}(e^{i\theta})}\, d\theta$$

$= J_0(2|c|t)$, where J_0 is the order-0 Bessel function of the first kind:

- shows up when computing cylindrical harmonics (solutions of Laplace's equation in cylindrical coords)
- $J_0(x)$ is a solution to

$$x^2y'' + xy' + x^2y = 0.$$



Fact: $\hat{\eta}(t) = J_0(2|c|t)$ is the characteristic function of the random variable $2\mathrm{Re}(cZ)$, where Z is uniform on unit circle.

More complicated example:

Let $0 < r_1 < r_2 < \cdots < r_n$. Define

$$E(x) = b + \sum_{j=1}^{n} 2\,\mathrm{Re}\left(c_j x^{ir_j}\right)$$

for constants $b \in \mathbb{R}$, $c_j \in \mathbb{C}$.

If we assume that $\{r_1, \dots, r_n\}$ is linearly independent over $\mathbb{Q}$, then by Kronecker–Weyl thm, the limiting log distribution of the

for $t(x^{ir_1}, x^{ir_2}, \dots, x^{ir_n})$

is Haar measure on $\mathbb{T}^n$.

So if f is bounded, continuous on $\mathbb{T}^n$, then $(y = \log x)$

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(e^{ir_1 y}, \dots, e^{ir_n y})\,dy$$

$$= \int_{\mathbb{T}^n} f(\vec{a})\, d\mathrm{Haar}(\vec{a}),$$

which is the same as the expectation of $f(Z_1, Z_2, \dots, Z_n)$ where Z_j are uniformly dist'd on S^1 and are <u>independent</u>.

~~Take $f(\vec{x}) = e^{itE(x)}$ to obtain~~

$E(x) = f(x^{ir_1}, \dots, x^{ir_n})$ where

$$f(\vec{z}) = b + \sum_{j=1}^{n} 2\,\mathrm{Re}(c_j z_j)$$

to obtain the Fourier transform of the limiting log dist'n of $E(x)$:

$$\left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} e^{it\left(b + 2\mathrm{Re}\left(\sum_{j=1}^{n} c_j e^{i\theta_j}\right)\right)}\, d\theta_1 \cdots d\theta_n$$

$$= e^{itb} \prod_{j=1}^{n} \frac{1}{2\pi}\int_0^{2\pi} e^{it\cdot 2\mathrm{Re}\, c_j e^{i\theta_j}}\, d\theta_j$$

Conclusion: the Fourier transform of the limiting log distr'n of $E(x) = b + \sum_{j=1}^{n} 2\operatorname{Re}(c_j x^{i r_j})$

is equal to

$$e^{itb} \prod_{j=1}^{n} J_0(2|c_j| t),$$

which is the characteristic function of the random variable

$$b + \sum_{j=1}^{n} 2\operatorname{Re}(c_j Z_j)$$

where Z_j are independent, uniform on S^1.