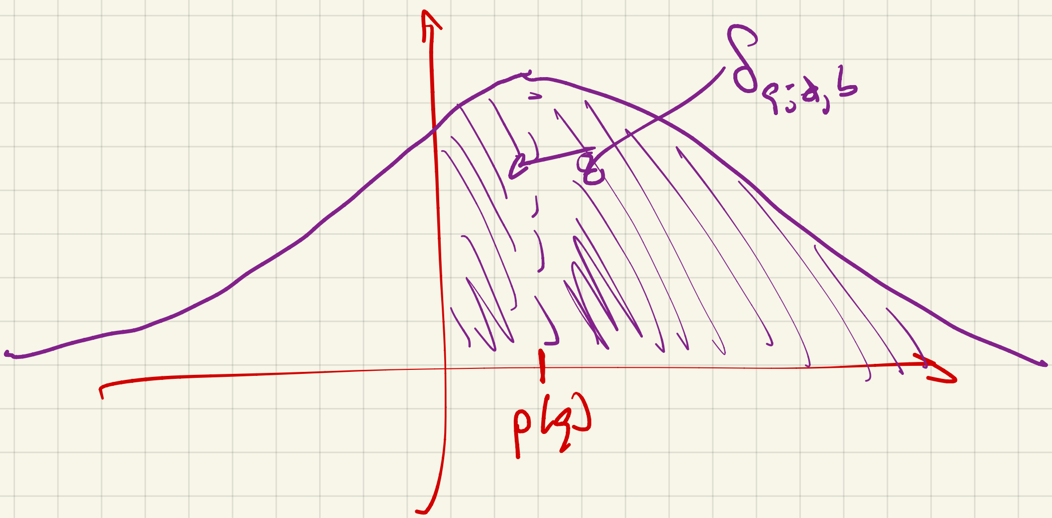
Friday, March 10

Notation reminder:

- $X_{q;a,b} = c(q,b) - c(q,a) + \sum_{\chi \,(\mathrm{mod}\, q)} |\chi(b)-\chi(a)| \sum_{\gamma>0} \frac{2\,\mathrm{Re}\, Z_\gamma}{\sqrt{1/4+\gamma^2}}$

where Z_γ independent, unif. dist'd on S^1.

- $\delta_{q;a,b} = \Pr(X_{q;a,b} > 0)$



- $V(q;a,b) = \sigma^2(X_{q;a,b})$

$$= \sum_{\chi \,(\mathrm{mod}\, q)} |\chi(b)-\chi(a)|^2\, b(\chi).$$

Today: investigate the dependence of $V(q;a,b)$ on a and b (under the standing assumptions $a \not\equiv \square$, $b \equiv \square$)

Initial observations:

- If $(r,q)=1$, then

$$|\chi(br)-\chi(ar)| = |\chi(r)(\chi(b)-\chi(a))| = |\chi(b)-\chi(a)|;$$

in particular, if $r \equiv \square \pmod q$ then

$$X_{q;ar,br} = X_{q;a,b}.$$

Thus (by choosing $r \equiv b^{-1} \pmod q$) we may restrict to considering $a \not\equiv \square$, $b \equiv 1 \pmod q$.

- Also note that

$$|\chi(1)-\chi(a^{-1})| = |1-\overline{\chi(a)}| = |\chi(1)-\chi(a)|;$$

so $X_{q;a^{-1},1} = X_{q;a,1}$.

- On GRH, there's an exact formula for $b(\chi)$, due to Vorhauer:

$$b(\chi) = \sum_{\substack{\gamma \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{1}{\frac{1}{4}+\gamma^2} \qquad \boxed{\chi \neq \chi_0}$$

$$= \log\left(\frac{q^*}{\pi}\right) - C_0 - (1+\chi(-1))\log 2 + 2\,\mathrm{Re}\,\frac{L'}{L}(1,\chi^*).$$

- C_0 is Euler's constant
- q^* is the conductor of χ.

$$C_0 = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j} - \log n\right) \doteq 0.577$$

$$\zeta(s) = \frac{1}{s-1} + C_0 + O(|s-1|)$$ near $s=1$.

Proposition 3.1 ("Inequities") Let a,b be distinct residue classes $(\mathrm{mod}\ q)$.

Then:

- $$\sum_{\chi\,(\mathrm{mod}\ q)} |\chi(b)-\chi(a)|^2 = 2\phi(q)$$
- If $c \not\equiv 1 \pmod q$, $(c,q)=1$, then

$$\sum_{\chi\,(\mathrm{mod}\ q)} |\chi(b)-\chi(a)|^2\chi(c) = -\phi(q)\left(\iota_q(cab^{-1}) + \iota_q(cba^{-1})\right),$$

where $\iota_q(r) = \begin{cases} 1, & \text{if } r\equiv 1 \pmod q, \\ 0, & \text{if } r\not\equiv 1 \pmod q. \end{cases}$

Theorem 1.4: Assume GRH. Let $a, b \in (\mathbb{Z}/q\mathbb{Z})^\times$ be distinct. Then

$$V(q;a,b) = 2\phi(q)\Big(\mathcal{L}(q) + K_q(a-b) + \iota_q(-ab^{-1})\log 2 + 2M^*(q;a,b)\Big)$$

where

- $$\mathcal{L}(q) = \log q - \sum_{p \mid q} \frac{\log p}{p-1} + \frac{\Lambda(q)}{\phi(q)} - (\gamma_0 + \log 2\pi)$$
- $$K_q(n) = \frac{\Lambda(q/(q,n))}{\phi(q/(q,n))} - \frac{\Lambda(q)}{\phi(q)} \geq 0$$
- $$M^*(q;a,b) = \sum_{\chi \,(\mathrm{mod}\, q)}' |\chi(b) - \chi(a)|^2 \frac{L'}{L}(1,\chi^*).$$

Notes:
- $\mathcal{L}(q) = \log q + O(\log\log q)$
- if q is prime, then $\mathcal{L}(q) = \log\left(\frac{q}{2\pi e^{\gamma_0}}\right)$

Theorem 1.7: Assume GRH. Let r_1 and r_2 be the least positive residues of ab^{-1} and ba^{-1} (mod q). Then

$$M^*(q;a,b) = \phi(q)\left(\frac{\Lambda(r_1)}{r_1} + \frac{\Lambda(r_2)}{r_2} + H(q;a,b) + O\left(\frac{\log^2 q}{q}\right)\right),$$

where:

- if $p^\nu \| q$, set $$h(q;p,r) = \frac{1}{\phi(p^\nu)} \frac{\log p}{p^{e(q;p,r)}}$$
  where $e(q;p,r)$ is the smallest positive integer k with $p^e \equiv r^{-1} \pmod{q/p^\nu}$ (if not, use the convention $e(q;p,r) = \infty$)
- $$H(q;a,b) = \sum_{p^\nu \| q} \left(h(q;p,ab^{-1}) + h(q;p,ba^{-1})\right)$$

- $H(q;a,b)$ arises from changing $\frac{L'}{L}(1,\chi^*)$ to $\frac{L'}{L}(1,\chi)$

Recall Theorem 1.1: on GRH and LI,

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi V(q;a,b)}} + O\left(q^{-3/2+\varepsilon}\right)$$

(when $a \neq \square$, $b = \square$ modulo q).

Taking the linear approximation to $f(x) = \frac{\rho(q)}{\sqrt{2\pi x}}$ at $x = \mathcal{L}(q)$ gives:

Corollary 1.9 (same assumptions):

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi\phi(q)\mathcal{L}(q)}}\left(1 - \frac{\Delta(q;a,b)}{2\mathcal{L}(q)} + O\left(\frac{1}{\log^2 q}\right)\right),$$

where $\Delta(q;a,b) = K_q(a-b) + \iota_q(-ab^{-1})\log 2 + \frac{\Lambda(r_1)}{r_1} + \frac{\Lambda(r_2)}{r_2} + H(q;a,b)$.

- $\Delta(q;a,b) \geq 0$ and $\Delta(q;a,b) \ll 1$.
- $\Delta(q;a,b) \neq 0$ when
  - $a \equiv -b \pmod q$
  - $a \equiv b$ modulo a large divisor of q
  - r_1 or r_2 is a small prime power
  - (something about H)

$\Delta(q;a,b) > 0$ means $\delta_{q;a,b}$ is smaller (asymptotically)