Monday, March 13

<u>Recall:</u> • $\delta_{q;a,b}$ = logarithmic density of $\{x>0 : \pi(x;q,a) > \pi(x;q,b)\}$

<u>Corollary 1.9</u> ("Inequities"): Assuming GRH & LI, and $a \not\equiv \square \pmod q$ and $b \equiv \square \pmod q$,

$$\delta_{q;a,b} = \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi\phi(q)\mathcal{L}(q)}}\left(1 - \frac{\Delta(q;a,b)}{\mathcal{L}(q)} + O\left(\frac{1}{\log^2 q}\right)\right),$$

where $\mathcal{L}(q) \sim \log q$ and

$$\Delta(q;a,b) = K_q(a-b) + \iota_q(-ab^{-1})\log 2 + \frac{\Delta(r_1)}{r_1} + \frac{\Delta(r_2)}{r_2} + H(q;a,b),$$

where

$$\iota_q(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod q, \\ 0, & \text{otherwise;} \end{cases}$$

$$K_q(n) = \frac{\Lambda(q/(q,n))}{\phi(q/(q,n))};$$

r_1, r_2 are the least positive residues of ab^{-1} and $ba^{-1} \pmod q$.

Recall: $0 \le \Delta(q;a,b) \ll 1$

• the larger $\Delta(q;a,b)$ is, the smaller $\delta_{q;a,b}$ is.

Theorem 1.10: Assume GRH + LI.

- Convention: whenever we write $\delta_{q;a,1}$, we implicitly restrict to q such that $(a,q)=1$ and $a \not\equiv 1 \pmod q$.

Fix integers a, a'.

(a) If $a \neq 1$, then $\delta_{q;-1,1} < \delta_{q;a,1}$ for all but finitely many q.

(b) If a is a prime power and $a' \neq -1$, $a' \neq$ prime power, then $\delta_{q;a,1} < \delta_{q;a',1}$ for all but finitely many q.

(c) If a, a' are prime powers with $\frac{\Lambda(a)}{a} > \frac{\Lambda(a')}{a'}$, then $\delta_{q;a,1} < \delta_{q;a',1}$ for all but finitely many q.

Bays–Hudson "mirror image phenomenon modulo 11":

Let $q \equiv 3 \pmod 4$ be a prime; we observe that if b is a square $\pmod q$, then when $\pi(x;q,b)$ is unusually small, $\pi(x;q,-b)$ is unusually large.

Let's investigate $E(x;q,b) + E(x;q,-b)$.

$E(x;q,b) + E(x;q,a)$

- under GRH and LI, has a limiting log'c distn with variance

$$V^+(q;a,b) = \sum_{\chi \pmod q} |\chi(b)+\chi(a)|^2 b(\chi).$$

Evaluating $\sum_{\chi \pmod q} |\chi(b) + \chi(a)|^2 \, \delta(\chi)$

(equation (4.5) in "Inequities") =

$$V^+(q;a,b) = 2\phi(q)\Big(\mathcal{L}^+(q) - K_q(a-b) \boxed{- c(-ab^{-1}) \log 2} + 2M^+(q;a,b) - 4\delta(\chi_0) \Big)$$

on

when $a \equiv -b \pmod q$, the variance is smaller!