Wednesday, March 15

Warm-up calculation: Recall

$$\psi(x,\chi) = \sum_{n\le x} \Lambda(n)\chi(n).$$

We can work out

$$\frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} |\psi(x,\chi)|^2$$

$$= \frac{1}{\phi(q)} \sum_{\chi\,(\mathrm{mod}\, q)} \left(\sum_{n\le x} \Lambda(n)\chi(n)\right)\overline{\left(\sum_{m\le x} \Lambda(m)\chi(m)\right)}$$

$$= \frac{1}{\phi(q)} \sum_{\substack{n\le x\\ m\le x}} \Lambda(n)\Lambda(m) \sum_{\chi\,(\mathrm{mod}\, q)} \chi(n)\overline{\chi}(m)$$

$$= \sum_{\substack{n\le x\\ m\le x\\ (nm,q)=1\\ n\equiv m\,(\mathrm{mod}\, q)}} \Lambda(n)\Lambda(m) =$$

$$= \sum_{\substack{a\,(\mathrm{mod}\, q)\\ (a,q)=1}} \sum_{\substack{n\le x\\ m\le x\\ n\equiv m\equiv a\,(\mathrm{mod}\, q)}} \Lambda(n)\Lambda(m)$$

$$= \sum_{\substack{a\,(\mathrm{mod}\, q)\\ (a,q)=1}} \left(\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\, q)}} \Lambda(n)\right)^2$$

$$= \sum_{\substack{a\,(\mathrm{mod}\, q)\\ (a,q)=1}} \psi(x;q,a)^2.$$

Exercises: Show that

$$V(x;q) = \frac{1}{\phi(q)} \sum_{\substack{\chi\,(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} |\psi(x,\chi)|^2$$

$$= \sum_{\substack{a\,(\mathrm{mod}\, q)\\ (a,q)=1}} \left|\psi(x;q,a) - \frac{1}{\phi(q)}\sum_{\substack{n\le x\\ (n,q)=1}} \Lambda(n)\right|^2$$

and also

$$G(x;q) = \sum_{(a,q)=1} \left| \theta(x;q,a) - \frac{x}{\phi(q)} \right|^2$$

$$= \frac{1}{\phi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \left| \theta(x,\chi) - \begin{cases} x, & \text{if } \chi = \chi_0, \\ 0, & \text{otherwise} \end{cases} \right|^2$$

Hooley (1970s) conjectured that as soon as q goes to infinity with x, we have

$$\left.\begin{matrix} G(x;q) \\ V(x;q) \end{matrix}\right\} \ll x \log q.$$

Note: we don't expect this upper bound when q is fixed.

- Littlewood (1910s):

$$\left.\begin{matrix} \pi(x) - li(x) \\ \pi(x;4,3) - \pi(x;4,1) \\ \pi(x;3,2) - \pi(x;3,1) \end{matrix}\right\} = \Omega_{\pm}\left(\frac{\sqrt{x}}{\log x} (\log\log\log x) \right)$$

which implies

$$\left.\begin{matrix} G(x;4) \\ V(x;4) \\ G(x;3) \\ V(x;3) \end{matrix}\right\} = \Omega\left(x (\log\log\log x)^2 \right)$$

which is not $\ll x$.

- Davidoff (1980s?) showed for fixed q,

$$\left.\begin{matrix} G(x;q) \\ V(x;q) \end{matrix}\right\} = \Omega\left(x (\log\log\log x)^2 \right).$$

not $\ll_q x$.

The average version,

$$(*) \quad \frac{1}{Q} \sum_{q \leq Q} V(x;q) \ll x \log Q,$$

is the Barban-Davenport-Halberstam Thm,

valid for $Q \geq x/(\log x)^A$.

$$V(x;q) = \frac{1}{\phi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} |\psi(x,\chi)|^2$$

- For individual q: if we assume GRH and a strong version of the prime pairs conjecture (Hardy–Littlewood), Friedlander–Goldston showed $V(x;q) \ll x \log q$ for $q \geq x^{\frac{1}{2}+\varepsilon}$.
- For $q < x^{\frac{1}{2}}$, nothing known (even conditionally).

Fiorilli (2015), "The distribution of the variance of primes in arithmetic progressions". conjectured:

$$V(x;q) \ll x \log q \text{ when } q > (\log\log x)^{1+\varepsilon}$$

but not when $q < (\log\log x)^{1-\varepsilon}$.

Heuristic/motivation:

There's some random variable H_q that models $V(x;q)$ (or $G(x;q)$).

(– recall that

$$\psi(x,\chi) = -\sum_{\substack{L(\rho,\chi)=0 \\ \text{nontrivial}}} \frac{x^\rho}{\rho} + \text{small.})$$

- $\mathbb{E}(H_q) \sim \phi(q) \log q$
- $\sigma^2(H_q) \sim 2\phi(q)(\log q)^2$
- H_q "roughly normal variable"

models $\phi(q) \frac{V(x;q)}{x}$

Fiorilli proved, about H_q, that

$$\frac{1}{4}e^{-c_1\varepsilon^2\phi(q)} \leq \Pr\left(|H(q)-\phi(q)\log q| > \varepsilon\phi(q)\log q\right) \leq 2e^{-c_2\varepsilon^2\phi(q)}$$

(for some $c_1, c_2 > 0$).

These are "tail estimates" for H_q

$\Rightarrow$ heuristically give conjectures for how large x needs to be for large deviations to occur.

$\Rightarrow$ Fiorilli's cutoff $\log\log x$

Fiorilli–M. (2023+) "Disproving Hooley's conjecture"

Theorem: $$\frac{G(x;q)}{V(x;q)} = \Omega\left(x \log q \, \frac{\log\log x}{q}\right) \quad (\star)$$

In particular, Hooley's conjecture fails to hold in the range $q < \delta \log\log x$ if δ is sufficiently small.

We proved $\star$ holds for a positive proportion of q.

- GRH is false: $\star$ holds for every multiple q of q_0, where GRH(q_0) is false
- GRH is true: extension of the Hardy–Littlewood / Davenport method (ALL q)