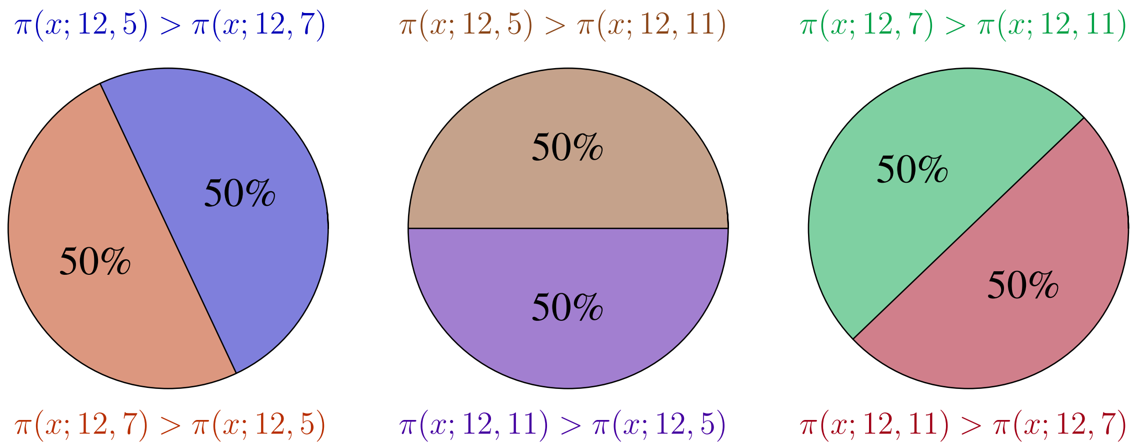
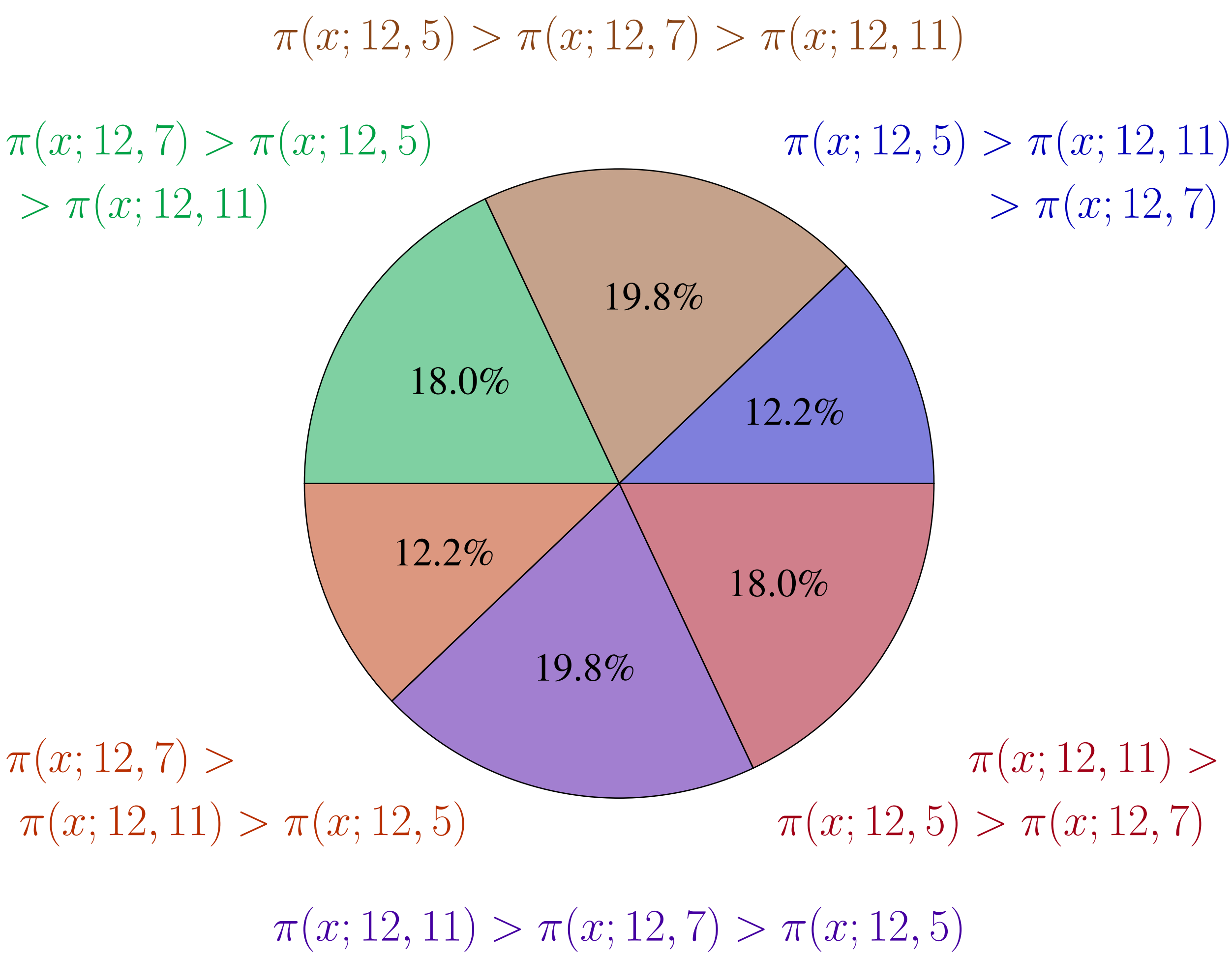
The three-way (mod 12) race among 5, 7, and 11:





Friday, March 17

Toy example in probability:

Let X_1, X_2, X_3 be independent random variable, uniform on $[-50,50]$, $[-8,6]$, and $[-6,6]$.

- Note that $\Pr(X_i > X_j) = \frac{1}{2}$ for any $i \neq j$, $i,j \in \{1,2,3\}$.

- Three-way orderings:
  - Suppose $X_1 > 6$ (44%).
    - $X_1 > X_2 > X_3$ (22%)
    - $X_1 > X_3 > X_2$ (22%)
  - Suppose $X_1 < -8$ (44%)
    - $X_2 > X_3 > X_1$ (22%)
    - $X_3 > X_2 > X_1$ (22%)
  - Suppose $X_1 \in [-6,8]$.
    - each of the 3! three-way orderings is equally likely: 2%.

So overall, the three-way races solves:

- $X_1 > X_2 > X_3$: 24%
- $X_1 > X_3 > X_2$: 24%
- $X_2 > X_3 > X_1$: 24%
- $X_3 > X_2 > X_1$: 24%
- $X_2 > X_1 > X_3$: 2%
- $X_3 > X_1 > X_2$: 2%

Next example: X_1, X_2, X_3 independent normal random variables with mean 0, variances $\alpha^2, \beta^2, \gamma^2$.

- Each X_j vs. X_i is 50/50 race.
- Symmetry: $\Pr(X_i > X_j > X_k) = \Pr(X_k > X_j > X_i)$.

Turns out we can calculate $\Pr(X_1 < X_2 < X_3)$.

- Density function of X_1 is $\frac{1}{\alpha\sqrt{2\pi}} e^{-x^2/2\alpha^2}$. (etc.)

We want to calculate $\Pr(X_1 < X_2 < X_3) =$

$$\int_{-\infty}^{\infty}\int_{x}^{\infty}\int_{y}^{\infty} \frac{1}{\alpha\beta\gamma(2\pi)^{3/2}} e^{-x^2/2\alpha^2 - y^2/2\beta^2 - z^2/2\gamma^2}\, dz\, dy\, dx.$$

Change variables $y = x+s$, $z = x+s+t$:

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\alpha\beta\gamma(2\pi)^{3/2}} e^{-\frac{x^2}{2\alpha^2} - \frac{(x+s)^2}{2\beta^2} - \frac{(x+s+t)^2}{2\gamma^2}}\, dt\, ds\, dx$$

Fact: $\int_{-\infty}^{\infty} e^{-ax^2 - bx}\, dx = \int_{-\infty}^{\infty} e^{-a\left(x - \frac{b}{2a}\right)^2 + \frac{b^2}{4a}}\, dx$

$$= e^{b^2/4a} \int_{-\infty}^{\infty} e^{-a\left(\underset{x}{\cancel{x - b/2a}}\right)^2} dx = e^{b^2/4a}\sqrt{\frac{\pi}{a}}.$$

Thus

$$\frac{1}{\alpha\beta\gamma(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2} - \frac{(x+s)^2}{2\beta^2} - \frac{(x+s+t)^2}{2\gamma^2}}\, dx =$$

$$= \frac{1}{2\pi\tau} \exp\left(-\frac{\overbrace{s^2(\beta^2+\gamma^2) - 2st(\alpha^2+\gamma^2) + t^2(\alpha^2+\beta^2)}}{2\tau^2}\right), \quad (*)$$

where $\tau^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$.

- Remaining integral is $\int_0^\infty\int_0^\infty \; dz\, dy$, which is an "orthant probability" — $\Pr$(some normal random variable lies in $[0,\infty)^2$).

— Possible to diagonalize the quadratic form $(*)$ to $e^{-(v^2+w^2)}$ by linear change of variables; region of integration becomes a sector in $\mathbb{R}^2$.

— trivial to integrate in polar coords.

Final answer:

$$\Pr(X_1 < X_2 < X_3) = \frac{1}{2\pi}\arctan\frac{\tau}{\beta^2}.$$

$$\Pr(X_1<X_2<X_3) = \frac{1}{2\pi}\arctan\left(\frac{\sqrt{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}}{\beta^2}\right)$$
(*)

If $\alpha^2, \beta^2, \gamma^2$ are all close to each other:

$$\alpha^2=\sigma^2(1+\varepsilon),\quad \beta^2=\sigma^2(1+\xi),\quad \gamma^2=\sigma^2(1+\eta).$$

Linear approximation to (*) at $(\varepsilon,\xi,\eta)=(0,0,0)$ is

$$\Pr(X_1<X_2<X_3) = \frac{1}{6} + \frac{\varepsilon-2\xi+\eta}{8\pi\sqrt{3}} + O(\varepsilon^2+\xi^2+\eta^2).$$

- Literature contains closed formulas for 3- and 4-dimensional orthant probabilities of multivariate normal random variable with mean 0 — even if nondiagonal ($X_1,\dots,X_4$ don't have to be independent)

– When we estimate $\delta_{q;a_1,\dots,a_r}$ = log'c density of $\{x : \pi(x;q,a_1)>\cdots>\pi(x;q,a_r)\}$ by $\Pr(X_1>\cdots>X_r)$, we get roughly

$\delta_{q;a_1,\dots,a_r}$ = (some orthant probability in $\mathbb{R}^r$) + $O\left(\frac{1}{q}\right)$.

Example: Lamzouri, "Prime number races w/ 3 or more competitors",

Corollary 2.3:

$$\delta_{q;a_1,a_2,a_3} = \frac{1}{6} + \frac{1}{4\sqrt{\pi}}\frac{c(q,a_3)-c(q,a_1)}{\sqrt{V(q)}} + \frac{1}{4\pi\sqrt{3}}\frac{B_q(a_1,a_2)-2B_q(a_1,a_3)+B_q(a_2,a_3)}{V(q)} + O(\text{blah}^2),$$

$$c(q,a) = -1+\#\{b^2\equiv a \pmod q\}$$

$$V(q) = 2\sideset{}{'}\sum_{\chi \,(\mathrm{mod}\, q)} \sum_{\substack{\gamma>0\\ L(\frac12+i\gamma,\chi)=0}} \frac{1}{\frac14+\gamma^2} \sim 2\phi(q)\log q$$

$$B_q(a,b) = 2\sideset{}{'}\sum_{\chi \,(\mathrm{mod}\, q)} \left(\chi(ab^{-1})+\chi(ba^{-1})\right)\sum_{\gamma>0}\frac{1}{\frac14+\gamma^2}.$$