Prime number races with three or more competitors

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Introduction and notations

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Rubinstein and Sarnak (1994)

Assume GRH and LI. Let $P_{q;a_1,\ldots,a_r}$ be the set of real numbers $x \geq 2$ such that

$$\pi(x; q, a_1) > \pi(x; q, a_2) > \cdots > \pi(x; q, a_r).$$

The logarithmic density of $P_{q;a_1,\ldots,a_r}$ defined by

$$\delta_{q;a_1,\ldots,a_r} := \lim_{x\to\infty} \frac{1}{\log x} \int_{t\in P_{q;a_1,\ldots,a_r}\cap[2,x]} \frac{dt}{t},$$

exists and is positive.

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- It follows from the work of Rubinstein and Sarnak that

$$\delta_{q;a_1,\dots,a_r} = \mathbb{P}(\mathbb{X}(q, a_1) > \mathbb{X}(q, a_2) > \cdots > \mathbb{X}(q, a_r)).$$

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where

$$\mathbb{X}(q,a) := -c(q,a) + \sum_{\substack{\chi \\ \chi\neq\chi_0 \\ \pmod q}} \mathrm{Re}\Big(2\chi(a)\sum_{\gamma_\chi>0}\frac{U(\gamma_\chi)}{\sqrt{\frac14+\gamma_\chi^2}}\Big),$$

and $c(q,a) := -1 + |\{n \pmod q : n^2 \equiv a \pmod q\}|$, and $\{U(\gamma_\chi)\}_{\gamma_\chi\in\Gamma}$ is a sequence of independent random variables uniformly distributed on the unit circle $\mathbb{S}^1$.

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- In the notation of G. Martin's notes, the random variable $X_{q;a,b}$ has the same distribution as $\mathbb{X}(q, a) - \mathbb{X}(q, b)$ defined above.

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$$\delta_{q;a,b} = \mathbb{P}(X_{q;a,b} > 0) = \mathbb{P}(\mathbb{X}(q,a) > \mathbb{X}(q,b)).$$

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$$\Delta_r(q) := \max_{(a_1,a_2,\ldots,a_r)\in\mathcal{A}_r(q)} \left|\delta_{q;a_1,\ldots,a_r} - \frac{1}{r!}\right| \to 0, \text{ as } q \to \infty.$$

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Ideas of the proof

- Show that the Fourier transform (properly normalized) of the joint distribution of the random vector $(\mathbb{X}(q,a_1),\ldots,\mathbb{X}(q,a_r))$ converges to the Fourier transform of a standard multivariate Gaussian vector $(Z_1,\ldots,Z_r)$ (i.e. the Z_j are I. I. D and $\sim \mathcal{N}(0,1)$).

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- By Levy's Continuity Theorem we deduce that

$$\delta_{q;a_1,\ldots,a_r} = \mathbb{P}(\mathbb{X}(q,a_1) > \cdots > \mathbb{X}(q,a_r)) \to \mathbb{P}(Z_1 > \cdots > Z_r) = \frac{1}{r!}.$$

Asymptotic formulas for the densities when $q \to \infty$

The case $r = 2$: Fiorilli and Martin (2013)

If a_1 is a non-square and a_2 is a square modulo q, then

$$\delta_{q;a_1,a_2} = \frac{1}{2} + \frac{c(q,a_2) - c(q,a_1)}{2\sqrt{\pi V(q)}}(1 + o(1)),$$

where

$$V(q) := 2 \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{1}{\frac{1}{4} + \gamma_\chi^2} \sim \varphi(q) \log q.$$

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Corollary (Fiorilli and Martin, 2013)

$$\Delta_2(q) = \frac{1}{q^{1/2+o(1)}}.$$

The case $r \geq 3$

- While the behaviour of the densities $\delta_{q;a_1,a_2}$ is governed by the **means** of the random variables $\mathbb{X}(q, a_1)$ and $\mathbb{X}(q, a_2)$, the behaviour of $\delta_{q;a_1,a_2,\dots,a_r}$ for $r \geq 3$ will be governed by the **correlations** of the $\mathbb{X}(q, a_j)$'s.

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Definition

The **covariance matrix** of the random vector $(X_1, \dots, X_r)$ is the $r \times r$ matrix K whose entries are

$$K_{i,j} = \mathbb{E}\big((X_i - \mathbb{E}(X_i))(X_j - \mathbb{E}(X_j))\big).$$

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In particular the diagonal entries of K are the variances of the X_j's, namely

$$K_{j,j} = \mathrm{Var}(X_j).$$

Exercise 1

Let $\mathcal{C} = \mathcal{C}_{q;a_1,\dots,a_r}$ be the covariance matrix of the random vector $\big(\mathbb{X}(q,a_1),\dots,\mathbb{X}(q,a_r)\big)$. Show

$$\mathcal{C}_{i,j} = \begin{cases} V(q) & \text{if } i = j \\ B_q(a_i,a_j) & \text{if } i \neq j, \end{cases}$$

where

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and

$$B_q(a,b) = \sum_{\substack{\chi \neq \chi_0 \\ \chi \bmod q}} \sum_{\gamma_\chi > 0} \frac{\chi\left(\frac{a}{b}\right) + \chi\left(\frac{b}{a}\right)}{\frac{1}{4} + \gamma_\chi^2}.$$

Proposition

For a, b such that $1 \leq |a|, |b| \leq q/2$ we have

$$B_q(a, b) = -\varphi(q)\left(\ell(a, b)\log 2 + \frac{\Lambda(s_1)}{s_1} + \frac{\Lambda(s_2)}{s_2}\right) + O\big((|a| + |b|)(\log q)^2\big),$$

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- In particular, we have $\max_{(a,b)\in\mathcal{A}_2(q)} |B_q(a, b)| \asymp \varphi(q)$, and hence

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- In particular, we have $\max_{(a,b)\in\mathcal{A}_2(q)} |B_q(a, b)| \asymp \varphi(q)$, and hence
$$\max_{(a,b)\in\mathcal{A}_2(q)} \frac{|B_q(a, b)|}{V(q)} \asymp \frac{1}{\log q}.$$
- However, we have $|B_q(a, b)| \asymp \log q$ on average over all $(a, b) \in \mathcal{A}_2(q)$.

Theorem 1 (L., 2013)

Let q be large. In the range $2 \leq r = o((\log q/(\log\log q))^{1/2}$, we have uniformly for all $(a_1, \ldots, a_r) \in \mathcal{A}_r(q)$

$$\delta_{q;a_1,\ldots,a_r} = \left(1 + O\left(\frac{r^4(\log r)^2}{(\log q)^2}\right)\right)\left(\frac{1}{r!} + \sum_{1\leq j<k\leq r} \beta_{j,k}(r)\frac{B_q(a_j, a_k)}{V(q)}\right),$$

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- Show that $\beta_{1,2}(2) = 0$ and for $r \ge 3$ that $\beta_{1,r}(r) < 0$ and $\beta_{r-1,r}(r) > 0$.

Consequences of the asymptotic formula

Recall that

$$\Delta_r(q) := \max_{(a_1,a_2,\ldots,a_r)\in\mathcal{A}_r(q)} \left|\delta_{q;a_1,\ldots,a_r} - \frac{1}{r!}\right|.$$

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$$\Delta_r(q) \asymp_r \frac{1}{\log q}.$$

Biased races

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Corollary 2 (L., 2013)

Fix $r \geq 3$. There exists a constant $q_0(r)$ such that if $q \geq q_0(r)$ is a positive integer, then

- There exist distinct residue classes $a_1, \ldots, a_r$ mod q, with $(a_i, q) = 1$, $a_1, \ldots, a_r$ are **squares** modulo q and the race $\{q; a_1, \ldots, a_r\}$ is biased.

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- There exist distinct residue classes $b_1, \ldots, b_r \bmod q$, with $(b_i, q) = 1$, $b_1, \ldots, b_r$ are **non-squares** modulo q and the race $\{q; b_1, \ldots, b_r\}$ is biased.

Explicit constructions

- Biased races with r squares:
  Let q be positive integer with $(q, 6) = 1$. Consider the race $\{q; 1, 6^4, 6^6, \dots, 6^{2(r-1)}, 4\}$. If q is large, then

$$\delta(q; 1, 6^4, 6^6, \dots, 6^{2(r-1)}, 4) > \frac{1}{r!} > \delta\left(q; 6^4, 6^6, \dots, 6^{2(r-1)}, 1, 4\right).$$

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- Biased races with r non-squares:
  Let $q \equiv 3 \bmod 4$ be a prime. Then -1 is a non-square modulo q. Consider the race $\{q; -1, -6^4, -6^6, \ldots, -6^{2(r-1)}, -4\}$. If q is large,

$$\delta(q; -1, \ldots, -6^{2(r-1)}, -4) > \frac{1}{r!} > \delta(q; -6^4, \ldots, -6^{2(r-1)}, -1, -4).$$

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- Let $\left(c_k(b_j)\right)_{\substack{1 \leq j \leq r \\ 1 \leq k \leq m}}$ be complex numbers.
- We consider the following vector of random variables $\mathbb{W} = (\mathbb{W}_1, \ldots, \mathbb{W}_r)$ where

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- Our goal to approximate the distribution of $\mathbb{W}$ by a multivariate Gaussian with the **same covariance matrix**, uniformly in all parameters.

Theorem (Reinert-Röllin (2009), Harper (2013))

Let $\mathbb{Y} = (\mathbb{Y}_1, \ldots, \mathbb{Y}_r)$ be a multivariate normal random vector with the **same covariance matrix** as $\mathbb{W} = (\mathbb{W}_1, \ldots, \mathbb{W}_r)$. Let $C := \max_{j,k} |c_k(b_j)|$ and assume that $\mathbb{E}\left(|\mathbb{V}_k|^4\right) \leq \frac{K^4}{m^2}$ for all $1 \leq k \leq m$ and some $K \geq 1$.

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$$|h|_2 := \sup_{1 \leq i,j \leq r} \left|\left| \frac{\partial^2}{\partial x_i \partial x_j} h \right|\right|_\infty, \text{ and } |h|_3 := \sup_{1 \leq i,j,k \leq r} \left|\left| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} h \right|\right|_\infty.$$

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Harper (2013) deduced this theorem from a general multivariate normal approximation result of Reinert and Röllin (2009), which they established using Stein's method of exchangeable pairs.

We will apply this result the random vector $\mathbb{W} = (\mathbb{W}_1, \ldots, \mathbb{W}_r)$ where

$$\mathbb{W}_j := \frac{\mathbb{X}(q, a_j) + c(q, a_j)}{\sqrt{V(q)}} = \sum_{\substack{\chi \neq \chi_0 \\ \chi \pmod q}} \mathrm{Re}\big(\chi(a_j)\mathbb{V}_\chi\big),$$

where

$$\mathbb{V}_\chi := \frac{2}{\sqrt{V(q)}} \sum_{\gamma_\chi > 0} \frac{U(\gamma_\chi)}{\sqrt{\frac{1}{4} + \gamma_\chi^2}},$$

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Exercise 3

Show that for any $\chi \neq \chi_0 \pmod q$ we have

$$\mathbb{E}\left(|\mathbb{V}_\chi|^4\right) \ll \frac{(\log q)^2}{V(q)^2} \ll \frac{1}{m^2}.$$

Hence we have $C = \max_{j,\chi} |\chi(a_j)| = 1$ and we can take K to be a fixed constant.

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Let $\mathbb{Y} = (\mathbb{Y}_1, \dots, \mathbb{Y}_r)$ denote a multivariate normal random vector whose components have mean zero, variance 1, and correlations

$$\mathbb{E}(\mathbb{Y}_j \mathbb{Y}_k) := \mathbb{E}(\mathbb{W}_j \mathbb{W}_k) = \frac{B_q(a_j, a_k)}{V(q)}.$$

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Then for any three times differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ we have

$$\left|\mathbb{E}(h(\mathbb{W})) - \mathbb{E}(h(\mathbb{Y}))\right| \ll \frac{r^2|h|_2 + r^3|h|_3}{\sqrt{\varphi(q)}}.$$

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Now we need to find a nice choice of the function h that approximates the characteristic function of the set $\{(x_1, \dots, x_r) \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_r\}$.

The choice of the function h

- Let $\delta > 0$ be a parameter to be chosen. Let $g : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function such that

$$g(x) = \begin{cases} 1 & \text{if } x \geq \delta, \\ \in [0,1] & \text{if } 0 < x \leq \delta, \\ 0 & \text{if } x \leq 0, \end{cases}$$

and such that $g^{(\ell)}(x) \ll (1/\delta)^{\ell}$ for $1 \leq \ell \leq 3$.

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and such that $g^{(\ell)}(x) \ll (1/\delta)^{\ell}$ for $1 \leq \ell \leq 3$.
- Note that such g exists since the interval on which g changes from 0 to 1 has length δ.
- Let $h_\delta^-, h_\delta^+ : \mathbb{R}^r \to \mathbb{R}$ be three times differentiable functions defined by
$$h_\delta^-(x_1, \dots, x_r) := \prod_{1 \leq i < j \leq r} g(x_i - x_j),$$
and
$$h_\delta^+(x_1, \dots, x_r) := \prod_{1 \leq i < j \leq r} g(x_i - x_j + \delta).$$

Exercise 4

a. Show that $\mathbb{E}\left(h_\delta^-(\mathbb{W})\right) \le \mathbb{P}(\mathbb{W}_1 > \mathbb{W}_2 > \cdots > \mathbb{W}_r) \le \mathbb{E}\left(h_\delta^+(\mathbb{W})\right)$, and that the same holds for $\mathbb{Y}$.

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b. Let $\delta_1, \ldots, \delta_r$ be such that $|\delta_j| \le \delta$. Show that

$$|\mathbb{P}(\mathbb{W}_1 + \delta_1 > \cdots > \mathbb{W}_r + \delta_r) - \mathbb{P}(\mathbb{W}_1 > \cdots > \mathbb{W}_r)| \ll r^2\delta,$$

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c. Use 1) and 2) to show that $\left|\mathbb{E}\left(h_\delta^\pm(\mathbb{W})\right) - \mathbb{P}(\mathbb{W}_1 > \mathbb{W}_2 > \cdots > \mathbb{W}_r)\right| \ll r^2\delta$, and that the same holds for $\mathbb{Y}$.

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d. Show that

$$|h_\delta^-|_2 \ll \frac{r^2}{\delta^2}, \text{ and } |h_\delta^-|_3 \ll \frac{r^3}{\delta^3},$$

and that the same bounds hold for h_δ^+.

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Let $\mathbb{Y} = (\mathbb{Y}_1, \ldots, \mathbb{Y}_r)$ denote a multivariate normal random vector whose components have mean zero, variance 1, and correlations

$$\mathbb{E}(\mathbb{Y}_j \mathbb{Y}_k) := \mathbb{E}(\mathbb{W}_j \mathbb{W}_k) = \frac{B_q(a_j, a_k)}{V(q)}.$$

Then we have

$$|\delta_{q;a_1,\ldots,a_r} - \mathbb{P}(\mathbb{Y}_1 > \cdots > \mathbb{Y}_r)| \ll \frac{r^3}{\varphi(q)^{1/8}}.$$

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Proof : First, recall that

$$\begin{aligned}\delta_{q;a_1,\dots,a_r} &= \mathbb{P}(\mathbb{X}(q,a_1) > \cdots > \mathbb{X}(q,a_r)) \\ &= \mathbb{P}\left(\frac{\mathbb{X}(q,a_1)}{\sqrt{V(q)}} > \cdots > \frac{\mathbb{X}(q,a_r)}{\sqrt{V(q)}}\right).\end{aligned}$$

- Moreover, we have

$$\mathbb{W}_j = \frac{\mathbb{X}(q, a_j) + c(q, a_j)}{\sqrt{V(q)}} = \frac{\mathbb{X}(q, a_j)}{\sqrt{V(q)}} + O(q^{-1/2+o(1)}).$$

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- Hence by Exercise 4b we deduce that

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- Furthermore, by Corollary 3 and Exercise 4d we have

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- We now use Exercise 4c to get

$$|\mathbb{P}(\mathbb{W}_1 > \cdots > \mathbb{W}_r) - \mathbb{P}(\mathbb{Y}_1 > \cdots > \mathbb{Y}_r)| \ll \frac{r^4/\delta^2 + r^6/\delta^3}{\sqrt{\varphi(q)}} + r^2\delta.$$

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- The result follows upon making the optimal choice $\delta = r/\varphi(q)^{1/8}$ and combining this estimate with (1).

The joint distribution of weakly correlated Gaussians

- Let $\mathbb{Y} = (\mathbb{Y}_1, \ldots, \mathbb{Y}_r)$ denote a multivariate normal random vector whose components have mean zero, variance 1, and correlations

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- Let $\mathcal{C} = (c_{j,k})_{1 \leq j,k \leq r}$ be the covariance matrix of $\mathbb{Y}$. Then $c_{j,j} = 1$ and $c_{j,k} = \mathbb{E}(\mathbb{Y}_j \mathbb{Y}_k) \ll \frac{1}{\log q}$, if $j \neq k$.

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- Let $\mathcal{C}^{-1} = (\tilde{c}_{j,k})_{1 \le j,k \le r}$. The joint density function of the random vector $\mathbb{Y}$ is given by

$$\begin{aligned} f(x_1, \dots, x_r) &= \frac{1}{(2\pi)^{r/2}\sqrt{\det(\mathcal{C})}} \exp\left(-\frac{1}{2}\mathbf{x}^T \mathcal{C}^{-1}\mathbf{x}\right) \\ &= \frac{1}{(2\pi)^{r/2}\sqrt{\det(\mathcal{C})}} \exp\left(-\frac{1}{2}\sum_{1 \le j,k \le r} \tilde{c}_{j,k} x_j x_k\right). \end{aligned}$$

Lemma (L., 2012)

Let $r \geq 2$ be an integer and $0 < \varepsilon \leq 1/(2r)$. Let $\mathcal{M}_r(\varepsilon)$ be the set of $r \times r$ symmetric matrices whose diagonal entries are 1, and whose off-diagonal entries have absolute value at most ε.

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b. A is invertible and if we denote by $\tilde{a}_{j,k}$ the entries of the inverse matrix A^{-1} then we have

$$\tilde{a}_{j,k} = \begin{cases} 1 + O(\varepsilon^2 r^2) & \text{if } j = k \\ -a_{j,k} + O(\varepsilon^2 r^2) & \text{if } j \neq k. \end{cases}$$

Proof : Exercise.

We have $\mathcal{C} \in \mathcal{M}_r(\varepsilon)$ where $\varepsilon = c/\log q$ for some constant $c > 0$.

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Hence we get

$$f(x_1,\dots,x_r) = \frac{1}{(2\pi)^{r/2}\sqrt{\det(\mathcal{C})}} \exp\left(-\frac{1}{2}\mathbf{x}^T\mathcal{C}^{-1}\mathbf{x}\right)$$
$$= \left(1+O\left(\frac{r^2}{(\log q)^2}\right)\right)\frac{1}{(2\pi)^{r/2}} \exp\left(-\frac{1}{2}\sum_{1\le j,k\le r}\tilde{c}_{j,k}x_jx_k\right)$$
$$= \left(1+O\left(\frac{r^2}{(\log q)^2}\right)\right)$$
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since $c_{j,k}=c_{k,j}$ and

$$\sum_{1\le j,k\le r}|x_jx_k| \le r\sum_{j=1}^{r}|x_j|^2 = r||\mathbf{x}||_2^2,$$

by the Cauchy-Schwarz inequality.

Recall that $c_{j,j} = \mathbb{E}(|Y_j|^2) = 1$ and $c_{j,k} = \mathbb{E}(\mathbb{Y}_j \mathbb{Y}_k) \ll \frac{1}{\log q}$, if $j \neq k$.

Recall that $c_{j,j} = \mathbb{E}(|Y_j|^2) = 1$ and $c_{j,k} = \mathbb{E}(\mathbb{Y}_j \mathbb{Y}_k) \ll \frac{1}{\log q}$, if $j \neq k$.

Exercise 5

If $r = o\big((\log q))^{2/3}\big)$ show that

- $$f(x_1, \ldots, x_r) \ll \frac{1}{(2\pi)^{r/2}} \exp\left(-\frac{||\mathbf{x}||_2^2}{4}\right).$$

Recall that $c_{j,j} = \mathbb{E}(|Y_j|^2) = 1$ and $c_{j,k} = \mathbb{E}(\mathbb{Y}_j \mathbb{Y}_k) \ll \frac{1}{\log q}$, if $j \neq k$.

Exercise 5

If $r = o\big((\log q))^{2/3}\big)$ show that

- $$f(x_1, \ldots, x_r) \ll \frac{1}{(2\pi)^{r/2}} \exp\left(-\frac{||\mathbf{x}||_2^2}{4}\right).$$
- Deduce that
$$\mathbb{P}(||\mathbb{Y}||_2 > R) \ll \exp\left(-\frac{R^2}{4} + O(r)\right).$$

We now have all the ingredients to prove our theorem.

Theorem 1 (L., 2013)

Let q be large. In the range $2 \le r = o((\log q/(\log\log q))^{1/2}$, we have uniformly for all $(a_1, \dots, a_r) \in \mathcal{A}_r(q)$

$$\delta_{q;a_1,\dots,a_r} = \left(1 + O\left(\frac{r^4(\log r)^2}{(\log q)^2}\right)\right)\left(\frac{1}{r!} + \sum_{1\le j<k\le r} \beta_{j,k}(r)\frac{B_q(a_j, a_k)}{V(q)}\right),$$

where

$$\beta_{j,k}(r) := \frac{1}{(2\pi)^{r/2}} \int_{x_1>\cdots>x_r} x_j x_k \exp\left(-\frac{x_1^2+\cdots+x_r^2}{2}\right) dx_1 \dots dx_r.$$

Proof of Theorem 1

By Corollary 4 we have

$$\left|\delta_{q;a_1,\dots,a_r} - \mathbb{P}(\mathbb{Y}_1 > \cdots > \mathbb{Y}_r)\right| \ll \frac{r^3}{\varphi(q)^{1/8}}.$$

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Let $R > c_0\sqrt{r}$ be a parameter to be chosen, where c_0 is a sufficiently large constant. By Exercise 5 we have

$$\begin{aligned}
&\mathbb{P}(\mathbb{Y}_1 > \cdots > \mathbb{Y}_r)\\
&= \mathbb{P}(\mathbb{Y}_1 > \cdots > \mathbb{Y}_r \text{ and } ||\mathbb{Y}||_2 \le R) + O\left(\exp\left(-\frac{R^2}{4} + O(r)\right)\right)\\
&= \int_{\substack{x_1 > \cdots > x_r\\ ||\mathbf{x}||_2 \le R}} f(x_1, \dots, x_r) dx_1 \cdots dx_r + O\Big(\exp\Big(-\frac{R^2}{5}\Big)\Big).
\end{aligned}$$

Now if $||\mathbf{x}||_2 \le R$ then we have

$$
\begin{aligned}
& f(x_1, \ldots, x_r) \\
& = \left(1 + O\left(\frac{r^2}{(\log q)^2}\right)\right) \\
& \quad \times \frac{1}{(2\pi)^{r/2}} \exp\Bigg(- \frac{||\mathbf{x}||_2^2}{2} + \sum_{1 \le j < k \le r} c_{j,k} x_j x_k + O\Big(\frac{r^3 ||\mathbf{x}||_2^2}{(\log q)^2}\Big) \\
& = \left(1 + O\left(\frac{r^3 R^2}{(\log q)^2}\right)\right) \frac{1}{(2\pi)^{r/2}} \exp\left(- \frac{||\mathbf{x}||_2^2}{2} + \sum_{1 \le j < k \le r} c_{j,k} x_j x_k \right)
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&= \left(1 + O\left(\frac{r^3 R^2}{(\log q)^2}\right)\right) \frac{1}{(2\pi)^{r/2}} \exp\left(-\frac{||\mathbf{x}||_2^2}{2} + \sum_{1 \leq j < k \leq r} c_{j,k} x_j x_k \right)
\end{aligned}$$

Moreover, we have

$$\exp\Big(\sum_{1 \leq j < k \leq r} c_{j,k} x_j x_k \Big) = 1 + \sum_{1 \leq j < k \leq r} c_{j,k} x_j x_k + O\left(\frac{r^2 ||\mathbf{x}||_2^4}{(\log q)^2} \right).$$

Therefore, we deduce that

$$f(x_1, \ldots, x_r) = \left(1 + O\left(\frac{r^2 R^4}{(\log q)^2}\right)\right) \frac{1}{(2\pi)^{r/2}} \exp\left(-\frac{||\mathbf{x}||_2^2}{2}\right)\left(1 + \sum_{1\le j<k\le r} c_{j,k} x_j x_k\right).$$

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To complete the proof we choose $R = c_1 \sqrt{r \log r}$ for some large constant $c_1 > 0$ and insert this last estimate in the asymptotic formula

$$\mathbb{P}(\mathbb{Y}_1 > \cdots > \mathbb{Y}_r) = \int_{\substack{x_1 > \cdots > x_r \\ ||\mathbf{x}||_2 \le R}} f(x_1, \dots, x_r) dx_1 \cdots dx_r + O\Big(\exp\left(-\frac{R^2}{5}\right)\Big).$$

Therefore, we deduce that

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Indeed the result follows upon completing the integrals and noting that

$$\frac{1}{(2\pi)^{r/2}}\int_{x_1>\cdots>x_r}\exp\left(-\frac{||\mathbf{x}||_2^2}{2}\right)dx_1\ldots dx_r = \frac{1}{r!},$$

and

$$\frac{1}{(2\pi)^{r/2}}\int_{x_1>\cdots>x_r}x_jx_k\exp\left(-\frac{||\mathbf{x}||_2^2}{2}\right)dx_1\ldots dx_r = \beta_{j,k}(r).$$

What happens if r is larger ?

Question (Feurverger and Martin, 2000)

- Is there a function $r_0(q) \to \infty$ as $q \to \infty$ such that for $r \geq r_0$ we have

$$\limsup_{q\to\infty} \max_{(a_1,\dots,a_r)\in\mathcal{A}_r(q)} r!\delta_{q;a_1,\dots,a_r} = \infty$$

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$$\liminf_{q\to\infty} \min_{(a_1,\dots,a_r)\in\mathcal{A}_r(q)} r!\delta_{q;a_1,\dots,a_r} = 0?$$

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- If so how quickly must $r_0(q)$ grow with q for these phenomena to emerge?

Conjecture (Ford and L., 2011)

1. If $2 \le r \le (\log q)^{1-\varepsilon}$, then

$$\lim_{q\to\infty} \max_{a_1,\dots,a_r \pmod q} |r!\delta(q; a_1, \dots, a_r) - 1| = 0.$$

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2. If $(\log q)^{1+\varepsilon} \leq r \leq \varphi(q)$, then

$$\lim_{q\to\infty} \max_{a_1,\ldots,a_r \pmod q} r!\delta(q; a_1, \ldots, a_r) = \infty,$$

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Theorem (Harper and L., 2018)

The first part of the Ford-Lamzouri Conjecture is true in the extended range $r = o\big((\log q)/(\log\log q)^4\big)$. More precisely, we have uniformly for $(a_1, \dots, a_r) \in \mathcal{A}_r(q)$

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Theorem (Ford, Harper and L., 2019)

The second part of the Ford-Lamzouri Conjecture is true as soon as $r/\log q \to \infty$. More precisely, in this range we have

$$\lim_{q\to\infty} \max_{a_1,\dots,a_r \pmod q} r!\delta(q; a_1, \dots, a_r) = \infty,$$

$$\lim_{q\to\infty} \min_{a_1,\dots,a_r \pmod q} r!\delta(q; a_1, \dots, a_r) = 0.$$

Thank you very much for your attention!