# Wednesday, March 22

Recall: $c(q,a) = -1 + \#\{b^2 \equiv a \pmod q\}$ and

$$E(x;q,a) = \frac{\phi(q)\pi(x;q,a) - \pi(x)}{\sqrt{x}/\log x}$$

$$= -c(q,a) + \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \bar{\chi}(a) \sum_{\substack{\gamma \in \mathbb{R} \\ L(\frac{1}{2}+i\gamma,\chi)=0}} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + o(1)$$

assuming GRH.

Remark: we can change $\pi(x)$ to $li(x)$, which changes $c(q,a)$ to $c(q,a)+1$, and it reintroduces $\chi_0$ into the sum.

Let's now look at the vector in $\mathbb{R}^r$:

$$(E(x;q,a_1), \ldots, E(x;q,a_r)) = \vec{b} + \sum_{\chi \neq \chi_0} \vec{c}_\chi \sum_{\gamma \in \mathbb{R}} \frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + o(1),$$

where

$\vec{b} = -(c(q,a_1), \ldots, c(q,a_r))$ and

$$\vec{c}_\chi = (\bar{\chi}(a_1), \ldots, \bar{\chi}(a_r)).$$

Assume GRH and LI throughout. Then this vector has a limiting logarithmic distribution, which is the same as the distribution of the random variable

$$\vec{b} + 2\,\mathrm{Re} \sum_{\chi \neq \chi_0} \vec{c}_\chi \sum_{\gamma > 0} \frac{Z_\gamma}{\sqrt{\frac{1}{4}+\gamma^2}},$$

where $Z_\gamma$ are independently uniformly distributed on the unit circle.

Remark: when $r=1$, we can write $\vec{c}_\chi Z_\gamma = \bar{\chi}(a) Z_\gamma \sim Z_\gamma$. However, we can't replace $\vec{c}_\chi$ by $(1,\ldots,1)$ when $r>1$ since the $\bar{\chi}(a_j)$ have different arguments.

Let $\mu_{q;\vec{a}} = \mu_{q;a_1,\ldots,a_r}$ be this limiting distribution.

The Fourier transform of $\mu_{q;\vec{a}}$ is

$$\hat{\mu}_{q;\vec{a}}(\vec{t}) = e^{-i\sum_{j=1}^{r} c(q,a_j)t_j} \times \prod_{\chi\neq\chi_0}\prod_{\gamma>0} J_0\left(\left|\frac{2}{\sqrt{\frac{1}{4}+\gamma^2}}\sum_{j=1}^{r}\chi(a_j)t_j\right|\right).$$

We want to understand the density

$$\delta_{q;\vec{a}} = \delta_{q;a_1,\dots,a_r} = \lim_{y\to\infty}\frac{1}{\log y}\int_1^y \mathbb{1}\big(\pi(x;q,a_1)>\dots>\pi(x;q,a_r)\big)\frac{dx}{x}$$

$$= \int\cdots\int_{x_1>\dots>x_r} d\mu_{q;\vec{a}}(\vec{x})\,;$$

using the change of variables $u_1=x_1-x_2$, $u_2=x_2-x_3$, $\dots$, $u_{r-1}=x_{r-1}-x_r$, $u_r=x_r$, and defining the corresponding measure $\nu_{q;\vec{a}}$ by $\mu_{q;\vec{a}}(x_1,\dots,x_r)=\nu_{q;\vec{a}}(x_1-x_2,\dots,x_{r-1}-x_r,x_r)$ we have

$$\delta_{q;\vec{a}} = \int_0^\infty\cdots\int_0^\infty\int_{-\infty}^\infty d\nu_{q;\vec{a}}(u_1,\dots,u_r)$$

$$= \int_0^\infty\cdots\int_0^\infty d\rho_{q;\vec{a}}(u_1,\dots,u_{r-1}),$$

where we've defined

$$\rho_{q;\vec{a}}(u_1,\dots,u_{r-1}) = \int_{u_r\in\mathbb{R}} d\nu_{q;\vec{a}}(u_1,\dots,u_{r-1},u_r).$$

Define $Q(u_1,\dots,u_{r-1}) = \mathbb{1}(u_1>0,\dots,u_{r-1}>0)$. Then by Plancherel,

$$\delta_{q;\vec{a}} = \int_{\mathbb{R}^{r-1}} Q(\vec{u})\,d\rho_{q;\vec{a}}(\vec{u})$$

$$= \int_{\mathbb{R}^{r-1}} \hat{Q}(\vec{t})\,\hat{\rho}_{q;\vec{a}}(\vec{t})\,d\vec{t}.$$

$$\delta_{q;\vec{a}} = \int_{\mathbb{R}^{r-1}} \hat{Q}(\vec{t})\, \hat{\rho}_{q;\vec{a}}(\vec{t})\, d\vec{t}$$

Formulas:

- $$\hat{\rho}_{q;\vec{a}}(t_1,\dots,t_{r-1}) = e^{-i\sum_{j=1}^{r-1}(c(q,a_j)-c(q,a_{j+1}))t_j}$$

$$\times \prod_{\chi\neq\chi_0}\prod_{\gamma>0} J_0\left(\frac{2}{\sqrt{\frac{1}{4}+\gamma^2}}\left|\sum_{j=1}^{r-1}(\chi(a_j)-\chi(a_{j+1}))t_j\right|\right)$$

- $$\hat{Q}(\vec{t}) = \prod_{j=1}^{r-1}\left(\frac{\delta(t_j)}{2} + \frac{i}{\pi t_j}\right).$$

- When $r=2$:

$$\delta_{q;a_1,a_2} = \frac{1}{2} + \frac{i}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \hat{\rho}_{q;a_1,a_2}(u)\,\frac{du}{u}$$

principal value

$$= \frac{1}{2} - \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sin((c(q,a_1)-c(q,a_2))u)}{u} \prod_{\chi}\prod_{\gamma}\cdots du$$

Reference: "Biases ...", Feuerverger – M.

- When $r=3$: write $\rho$ for $\rho_{q;a_1,a_2,a_3}$

$$\delta_{q;a_1,a_2,a_3} = \frac{1}{4} + \frac{i}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}} (\hat{\rho}(u,0)+\hat{\rho}(0,u))\,\frac{du}{u} + \frac{i^2}{4\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} \hat{\rho}(u,v)\,\frac{du}{u}\,\frac{dv}{v}.$$

- When $r=4$: write $\rho$ for $\rho_{q;a_1,\dots,a_4}$

$$\delta_{q;a_1,\dots,a_4} = \frac{1}{8} + \frac{i}{8\pi}\,\mathrm{PV}\int_{\mathbb{R}} (\hat{\rho}(u,0,0)+\hat{\rho}(0,u,0)+\hat{\rho}(0,0,u))\,\frac{du}{u}$$

$$+ \frac{i^2}{8\pi^2}\,\mathrm{PV}\int_{\mathbb{R}^2} (\hat{\rho}(u,v,0)+\hat{\rho}(u,0,v)+\hat{\rho}(0,u,v))\,\frac{du}{u}\,\frac{dv}{v}$$

$$+ \frac{i^3}{8\pi^3}\,\mathrm{PV}\int_{\mathbb{R}^3} \hat{\rho}(u,v,w)\,\frac{du}{u}\,\frac{dv}{v}\,\frac{dw}{w}.$$