Friday, March 24

For today: don't assume GRH!

Define $f(\rho, x) = \int_2^x \frac{t^\rho}{\rho} \, d\,\mathrm{li}(t)$

$$= \frac{x^\rho}{\rho \log x} + \frac{1}{\rho} \int_2^x \frac{t^\rho}{t \log^2 t} \, dt$$

Then for any $\beta_0 \geq \frac{1}{2}$,

$$\phi(q)\pi(x;q,a) - \pi(x) = -\sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \bar{\chi}(a) \sum_{\substack{L(\rho,\chi)=0 \\ \mathrm{Re}\,\rho > \beta_0}} f(\rho, x) + O_q(x^{\beta_0} \log x).$$

Hypothetical example:

(Granville, based on work of Ford and Konyagin): Let χ_5 be a complex char. (mod 5):

n	1	2	3	4	5
$\chi_5(n)$	1	i	$-i$	-1	0

Suppose that all zeros of the four $L(s,\chi)$ modulo 5 are on the critical line, except that $L(s,\chi_5)$ has a single zero $\rho_5 = \beta + i\gamma$ with $\beta > \frac{1}{2}$ (and thus $L(s, \bar{\chi}_5)$ has a zero at $\bar{\rho}_5 = \beta - i\gamma$). Then for any $\frac{1}{2} < \beta_0 < \beta$,

$$\phi(5)\pi(x;5,a) - \pi(x) = -\left(\bar{\chi}_5(a) f(\rho_5, x) + \chi_5(a) f(\bar{\rho}_5, x)\right) + O(x^{\beta_0} \log x).$$

Define $E_5(x;5,a) = \frac{\beta^2+\gamma^2}{8} \cdot \frac{4\pi(x;5,a) - \pi(x)}{x^\beta / \log x}$.

Then:

$$E_5(x;5,1) = -\beta\cos(\gamma\log x) - \gamma\sin(\gamma\log x) + o(1)$$
$$E_5(x;5,2) = +\beta\sin(\gamma\log x) - \gamma\cos(\gamma\log x) + o(1)$$
$$E_5(x;5,3) = -\beta\sin(\gamma\log x) + \gamma\cos(\gamma\log x) + o(1)$$
$$E_5(x;5,4) = +\beta\cos(\gamma\log x) + \gamma\sin(\gamma\log x) + o(1)$$

Claim: The chain of inequalities

$$\pi(x;5,1) \underset{a}{>} \pi(x;5,4) \underset{b}{>} \pi(x;5,2) \underset{c}{>} \pi(x;5,3) \quad (*)$$

is impossible when x is sufficiently large!

Proof: Suppose x satisfies $(*)$.

(a) $\Rightarrow \beta\cos(\gamma\log x) + \gamma\sin(\gamma\log x) < 0 + o(1)$

(c) $\Rightarrow \beta\sin(\gamma\log x) - \gamma\cos(\gamma\log x) > 0 + o(1)$

But then (b) implies

(d) $\beta\cos(\gamma\log x) + \gamma\sin(\gamma\log x) = o(1)$

(e) $\beta\sin(\gamma\log x) - \gamma\cos(\gamma\log x) = o(1)$.

Multiply (d) by $\cos(\gamma\log x)$ and (e) by $\sin(\gamma\log x)$ and add:

$$\beta = \beta\cos^2(\gamma\log x) + \beta\sin^2(\gamma\log x) = o(1)$$

which is a contradiction if x is sufficiently large.

Terminology (introduced by M.–Ng, "Inclusive prime number races"):

Let $a_1, \dots, a_r$ be distinct reduced residue classes (mod q). We say the race among the $\pi(x;q,a_j)$ is:

- exhaustive if, for each permutation $(\sigma_1, \dots, \sigma_r)$ of $(a_1, \dots, a_r)$, the system of inequalities

$$\pi(x;q,\sigma_1) > \dots > \pi(x;q,\sigma_r) \quad (**)$$

has arbitrarily large solutions x.

- **inclusive** if the real numbers x satisfying (**) has positive logarithmic density.
- **strongly inclusive** if the limiting log'c distribution of

$$(E(x;q,a_1), \ldots, E(x;q,a_r)) \in \mathbb{R}^r$$

has full support (gives positive mass to every open set in $\mathbb{R}^r$). (* for $r=\phi(q)$)

- Rubinstein and Sarnak proved that GRH and LI imply every r-way race is strongly inclusive.

<u>Definition</u>: A <u>barrier</u> is a hypothetical configuration of (pairs of) zeros of the $L(s,\chi)$, $\chi \pmod q$, $\chi \neq \chi_0$ that would force an r-way prime number race to not be exhaustive.

- Our first hypothetical example shows there exists a barrier of size 1 for the 4-way race (mod 5).

<u>Theorems</u> (Ford/Konyagin):
For every 3-way race, there exists a finite barrier; for most races, there exists a barrier of size 3. (Note that this implies the same for r-way races with $r \geq 4$.) Also, they can make these zeros arbitrarily close to the critical line and arbitrarily far from the real axis.