Function field and number field analogues of prime number races

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Analytic Number Theory II
Guest lecture

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If $(a, m) = 1$, then $\pi(N; m, a) = (\varphi(m))^{-1} \cdot \frac{q^N}{N} + O\left(\frac{q^{N/2}}{N}\right)$. Here $\varphi(m) = \#(\mathbb{F}_q[T]/(m))^{\times}$.

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Idea : factorize $T^{q^N} - T$ over $\mathbb{F}_q[T]$ and use Möbius inversion.

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Dirichlet L-functions for $\mathbb{F}_q[T]$

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Remarkable feature : let $u := q^{-s}$; one has

- if $\chi \neq \chi_0$, then $L(s,\chi) = \prod_{j=1}^{d(\chi)}(1 - \gamma_j(\chi)u)$, $|\gamma_j(\chi)| \in \{1, \sqrt{q}\}$,
- $L(s,\chi_0) = \frac{\prod_{P|m} 1-u^{\deg P}}{1-qu}$.

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$$u\frac{\mathrm{d}}{\mathrm{d}u} \log \mathcal{L}(u,\chi) = \sum_{N \geq 1} c_N(\chi)u^N ,$$

where $c_N(\chi) = \sum_{d|N} \sum_{P \nmid m, \deg P = d} \chi(P^{N/d})$.

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Also : $\mathcal{L}(u,\chi) = \prod_{j=1}^{d(\chi)} (1 - \alpha_j(\chi)u)$, $(\chi \neq \chi_0)$,

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$\mathcal{L}(u,\chi_0) = \frac{\prod_{P|m} 1-u^{\deg P}}{1-qu}$. Therefore for $\chi\neq\chi_0$:

$$c_N(\chi) = -\sum_{j=1}^{d(\chi)}\gamma_j(\chi)^N, \qquad c_N(\chi_0) = q^N + O(1)\,.$$

Chebyshev Bias in fct fields : relevance of squares

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Indeed :

$$\frac{1}{N} \sum_{\chi} \overline{\chi(a)} c_N(\chi) - \varphi(m)\pi(N; m, a) = \begin{cases} \sigma(m, a) \frac{q^{N/2}}{N} + O\Big(\frac{q^{N/3}}{N}\Big) \text{ } (N \text{ even}), \\ O\Big(\frac{q^{N/3}}{N}\Big) \text{ } (N \text{ odd}) . \end{cases}$$

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Here, reminiscent of Rubinstein–Sarnak's analysis, we define :

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$$\sigma(m, a) = \sum_{\substack{b \in (\mathbb{F}_q[T]/(m))^\times \\ b^2 \equiv a \bmod m}} 1 .$$

However, $\mathcal{L}(u, \chi)$ being algebraic,

$$\sum_{\chi} \overline{\chi(a)} c_N(\chi) = q^N + \sum_{\chi \neq \chi_0} \overline{\chi(a)} \sum_{j=1}^{d(\chi)} \gamma_j(\chi)^N + O(1) .$$

Function fields : explicit formula

Proposition (Cha, '08)

Let $\mathcal{B}(m,a,N) = \begin{cases} \sigma(m,a) - 1 \ (N \text{ even}) \\ 0 \ (N \text{ odd}) \end{cases}$, then :

$$\varphi(m)\pi(N;m,a) - \pi(N) = -\mathcal{B}(m,a,N)\frac{q^{N/2}}{N} - \frac{1}{N}\sum_{\chi \neq \chi_0} \overline{\chi(a)} \sum_{j=1}^{d(\chi)} \alpha_j(\chi)^N + O\Big(\frac{q^{N/3}}{N}\Big).$$

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$$E_{m,a}(X) = (1-\sigma(m,a))\mathcal{B}_q(X) - \sum_{\chi\neq\chi_0}\overline{\chi(a)}e^{i\theta(\gamma_\chi)}\frac{\gamma_\chi}{\gamma_\chi - 1} + o(1).$$

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Fix $a_1, \ldots, a_r$ invertible residue classes mod m. Consider the main term in the asymp. exp. of $E_{m,a_i}(X)$:

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$$E^{\mathrm{main}}_{m,a_i}(X) = (1 - \sigma(m,a))\mathcal{B}_q(X) - \sum_{\chi \neq \chi_0} \overline{\chi(a)} \mathrm{e}^{i\theta(\gamma_\chi)} \frac{\gamma_\chi}{\gamma_\chi - 1} .$$

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Proposition (Cha, '08)

For all continuous bounded f on $\mathbb{R}^r$, the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} \sum_{X=1}^{n} f((E^{\mathrm{main}}_{(a_i)}(X)) \,.$$

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Theorem (Cha, '08)

There exists a Borel prob. measure $\mu = \mu_{m,(a_i)}$ on $\mathbb{R}^r$ s.t., denoting $E_{m,(a_i)}(X) = (E_{m,a_1}(X), \dots, E_{m,a_r}(X))$, one has for any bounded continuous f on $\mathbb{R}^r$:

$$\int_{\mathbb{R}^r} f \mathrm{d}\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{X=1}^{n} f(E_{m,(a_i)}(X)) .$$

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Linear Independence (LI) hypothesis

As χ runs over non principal char.'s mod. m, the multiset

$$\{\theta \in [0,\pi] : \gamma = \sqrt{q}\mathrm{e}^{i\theta} \text{ inverse zero of } \mathcal{L}(u,\chi)\} \cup \{2\pi\}$$

is $\mathbb{Q}$-linearly independent.

LI and limiting distribution

Theorem (Cha, '08)

Assume LI ; then one computes the Fourier transform of μ :

$$\hat{\mu}(\xi) = \mathcal{B}_{m,(a_i)}(\xi) \prod_{\chi \neq \chi_0} \prod_{\mathrm{Im}(\gamma_\chi) > 0} J_0\Big(\frac{2|\gamma_\chi|}{|\gamma_\chi - 1|} \Big| \sum_{i=1}^{r} \chi(a_i)\xi_i \Big| \Big),$$

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where J_0 is the Bessel function and

$$\mathcal{B}_{m,(a_i)}(\xi) = \frac{1}{2}\exp\Big(i\frac{\sqrt{q}}{q-1}\sum_{i=1}^{r}\big(\sigma(m,a_i)-1\big)\xi_i\Big) + \frac{1}{2}\exp\Big(i\frac{q}{q-1}\sum_{i=1}^{r}\big(\sigma(m,a_i)-1\big)\xi_i\Big).$$

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The factor $\mathcal{B}_{m,(a_i)}(\xi)$ is "responsible for the bias".

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where $\mathrm{res}^{\pm}(N) = \#\{P \in \mathbb{F}_q[T] : \chi_{\text{quad}}(P) = \pm 1, \deg P = N\}$.

Function fields : “residues vs non-residues”

Theorem (Cha ’08)

$$E_{m;\mathrm{quad}}(X) = -\mathcal{B}_q(X) - 2\sum_{j=1}^{k} \mathrm{Re}\left(\mathrm{e}^{i\theta_j X} \frac{\gamma_j}{\gamma_j - 1}\right) + o(1)\,,$$

where the γ_j's are the inverse zeros of $\mathcal{L}(u, \chi_{\mathrm{quad}})$ and we recall

$$\mathcal{B}_q(X) = \begin{cases} \sqrt{q}/(q-1) \ (X \text{ odd}) \\ q/(q-1) \ (X \text{ even}) \end{cases}$$

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Assume m is squarefree of odd degree $2g+1$.

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$$E_{m;\mathrm{quad}}(X) = -\mathcal{B}_q(X) - 2\sum_{j=1}^{k} \mathrm{Re}\left(e^{i\theta_j X}\frac{\gamma_j}{\gamma_j - 1}\right) + o(1)\,,$$

where the $\gamma_j' s$ are the inverse zeros of $\mathcal{L}(u,\chi_{\mathrm{quad}})$ and we recall

$$\mathcal{B}_q(X) = \begin{cases} \sqrt{q}/(q-1)\ (X \text{ odd}) \\ q/(q-1)\ (X \text{ even}) \end{cases}$$

Assume m is squarefree of odd degree $2g+1$.

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Family considered : Fix $f(T) \in \mathbb{F}_q[T]$ monic sqf of degree $2g$. For each $a \in U(\mathbb{F}_q) = \mathbb{F}_q \setminus \{\text{zeros of } f\}$, one sets

$$m_a(T) = (T-a)f(T)\,, \qquad C_a\colon\ Y^2 = m_a(X)\,.$$

“Sieve for Frobenius” : generic LI for χ_{quad}

Let $\mathcal{L}_a(u)$ be the numerator of the zeta function of $C_a/\mathbb{F}_q$ *i.e.* $\mathcal{L}_a(u) = \mathcal{L}(u, \chi_{\mathrm{quad}})$ relative to $\mathbb{F}_q[T]/(m_a(T))$.

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$$\frac{1}{|U(\mathbb{F}_q)|}\#\{a \in U(\mathbb{F}_q) : \text{ LI does not hold for } \mathcal{L}_a(u)\} \ll_g q^{-1/(2A)}(\log q)^{1-\delta}$$

where $A = 2g^2 + g + 2$ and $1 \geq \delta \sim (8g)^{-1}$ $(g \to \infty)$.

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Consequence (Cha, '08)

The limiting distribution $\mu_{m_a,\mathrm{quad}}$ associated with $E_{m_a,\mathrm{quad}}$ exists, and except for a proportion $\ll_g q^{-1/(2A)}(\log q)^{1-\delta}$ of $a \in U(\mathbb{F}_q)$, one has $\mu_{m_a,\mathrm{quad}}((-\infty, 0]) > \frac{1}{2}$.

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(Chebotarev for the normal closure of $\mathbb{Q}(2^{\frac{1}{3}})/\mathbb{Q}$ (group S_3).)

Chebotarev

- L/K gal. ext. of number fields ; $G := \mathrm{Gal}(L/K)$.
- $\mathfrak{p} \subset \mathcal{O}_K$ an unram. ideal in L/K.
- $\mathrm{Frob}_\mathfrak{p} \subset \mathrm{Gal}(L/K)$ the Frobenius conj. class at $\mathfrak{p}$ (lifts to G the Frobenius aut. on the level of residual fields $x \mapsto x^{\mathcal{N}\mathfrak{p}}$).

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Let $C \subset \mathrm{Gal}(L/K)$ be a conj. class, then

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"Chebotarev race" : let C_1, C_2 be conj. classes of G. Investigate the log-density of

$$\left\{x \geq 2 : \frac{|G|}{|C_1|}\pi_{L/K}(x; C_1) > \frac{|G|}{|C_2|}\pi_{L/K}(x; C_2)\right\}$$

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Hypotheses : GRH can be stated for $L(s, L/K, \chi)$; analytic continuation is conjectural as well.

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- $L = \mathbb{Q}(\theta)$; $\theta =$ complex root of $x^8 - 205x^6 + 13940x^4 - 378225x^2 + 3404025$.
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$$\psi(x; L/K, t) := \sum_{\substack{\mathfrak{p}\triangleleft\mathcal{O}_K \\ k\geq 1}} t(\mathrm{Frob}_{\mathfrak{p}}^k) \log(\mathcal{N}\mathfrak{p}) \mathbf{1}_{\mathcal{N}\mathfrak{p}^k\leq x}$$

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- $\mathrm{Ind}_G^{G^+}(\frac{|G|}{|C|}\mathbf{1}_C) = \frac{|G^+|}{|C^+|}\mathbf{1}_{C^+}$, where $G \leqslant G^+$ and C^+ the conj. class of G^+ containing the G-conj. class C.

Unconditional Chebyshev bias in number fields

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Lemma (Fiorilli–J.)

Same notation ; $r_G(g) := \#\{h \in G\colon h^2 = g\}$. One has :

$$\pi_{L/K}(x; t) = \int_2^x \frac{\mathrm{d}\psi(u; L/K, t)}{\log u} - \langle t, r_G \rangle \frac{\sqrt{x}}{\log x} + o\Big(\frac{\sqrt{x}}{\log x}\Big)\,.$$

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- $r_G(g_1) = 0$ and $r_G(g_2) = 4$, thus $\langle t, r_G\rangle \neq 0$,
- g_1 and g_2 are conjugate in S_8 and so $t^+ = 0$.

Theorem (Fiorilli–J., '22)

Unconditionally, there exists infinitely many Galois ext. of nb fields L/K and conj. classes of same size C_1, C_2 in $\mathrm{Gal}(L/K)$ giving rise to an *extreme bias*; precisely, for x big enough, $\pi_{L/K}(x;C_1) > \pi_{L/K}(x;C_2)$.

Thanks for your attention !