Wednesday, March 29

Setting: r-way prime race among $\pi(x;q,a_1), \ldots, \pi(x;q,a_r)$. For any permutation $(\sigma_1, \ldots, \sigma_r)$ of $(a_1, \ldots, a_r)$, consider

$$(*)\ \{x>0 : \pi(x;q,\sigma_1) > \cdots > \pi(x;q,\sigma_r)\}.$$

Definition: We say this prime number race is:

- exhaustive if $(*)$ is unbounded above for every permutation;
- inclusive if the logarithmic density of $(*)$ exists and is strictly positive;
- strongly inclusive if the limiting logarithmic distribution of $(E(x;q,a_1), \ldots, E(x;q,a_r))$ has full support on $\mathbb{R}^r$.

Note: strongly inclusive $\Rightarrow$ inclusive $\Rightarrow$ exhaustive

New addition: We say the race is weakly inclusive if the logarithmic density of $(*)$ exists (but may be 0).

— Morally speaking, "weakly inclusive" means the distribution is "nice" — absolutely continuous WRT Lebesgue measure suffices.

— Inclusive $\Rightarrow$ weakly inclusive; but neither of "exhaustive" and "weakly inclusive" implies the other.

- Rubinstein/Sarnak proved: GRH + LI imply all races are strongly inclusive.
- On the other hand, Ford/Konyagin/Lamzouri ("barriers") showed that 3-way races might not be exhaustive if GRH is false.

Goal for today: discuss work of M.–Ng ("Inclusive prime number races") that's between Rubinstein/Sarnak and Ford/Konyagin/Lamzouri.

Throughout, assume GRH.

Notation: let $\Gamma(\chi) = \{\gamma > 0 : L(\tfrac{1}{2}+i\gamma, \chi) = 0\}$

and $\Gamma(q) = \bigcup_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0}} \Gamma(\chi)$.

Definition: If $\gamma \in \Gamma(q)$, we say γ is self-sufficient iff $\gamma \notin \mathrm{Span}_{\mathbb{Q}}(\Gamma(q) \setminus \{\gamma\})$.

Define $\Gamma^S(\chi) = \{\gamma \in \Gamma(\chi) : \gamma \text{ is self-suff.}\}$

and $\Gamma^S(q) = \bigcup_{\chi \neq \chi_0} \Gamma^S(\chi)$.

First: two-way races between $\pi(x;q,a)$ and $\pi(x;q,b)$.

Th^{m} (M.–Ng)

- If the $L(s,\chi)$ for which $\chi(a) \neq \chi(b)$ collectively have ≥ 3 self-sufficient zeros, then the 2-way race is weakly inclusive.
- There exists a constant $W(q)$ such that if $\displaystyle\sum_{\chi(a)\neq\chi(b)} \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma} \geq W(q)$, then the 2-way race is inclusive.
- If $\displaystyle\sum_{\chi(a)\neq\pm\chi(b)} \sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma}$ diverges, then the 2-way race is strongly inclusive.

Note: $\Gamma(q)$ has $\sim \frac{\phi(q)}{2\pi} T \log T$ elements up to height T; but even if $\Gamma^S(\chi)$ had $\varepsilon T/\log T$ ordinates, $\sum \frac{1}{\gamma}$ still diverges.

Now results for big multi-way races:

Thm: • If every nonprincipal $L(s,\chi)$ has $\geq 2\phi(q)+1$ self-sufficient zeros, then every r-way race (mod q) is weakly inclusive.

- Devin improved $2\phi(q)+1$ to $\phi(q)$.

• There exists $W(q)$ such that if $\sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma} \geq W(q)$ for every nonprincipal χ (mod q), then every race (mod q) is inclusive.

• If $\sum_{\gamma \in \Gamma^S(\chi)} \frac{1}{\gamma}$ diverges for every nonprincipal χ (mod q), the every race (mod q) is strongly inclusive.

For a particular r-way race with $2<r<\phi(q)$, we need "enough" characters with lots of self-sufficient zeros.

Thm Suppose

$$\left\{ (\chi(a_1), \dots, \chi(a_r)) : \chi \neq \chi_0,\ \sum_{\gamma \in \Gamma^S(q)} \frac{1}{\gamma} \text{ diverges} \right\}$$

(χ (mod q))

spans $\mathbb{C}^r$. Then this r-way race is strongly inclusive.

General strategy in "Inclusive ...":

- split up the explicit formula into self-sufficient part and other part; $E(x;q,\alpha) = E^S(x;q,\alpha) + E^N(x;q,\alpha)$.
- use the full Kronecker-Weyl theorem to show: distribution of E is the same as random variable $\vec{X} = \vec{X}^S + \vec{X}^N$, where $\vec{X}^S$ and $\vec{X}^N$ are independent.
- Thus, distribution is a convolution $\mu = \mu^S * \mu^N$, with characteristic functions $\hat{\mu} = \hat{\mu}^S \cdot \hat{\mu}^N$.
- We know basically nothing about μ^N!

- However, μ^S has a familiar shape because all $\Gamma^S(q)$ is linearly independent. (LI holds for $\Gamma^S(q)$)
- The support of μ contains a translation of the support of μ^S.