Friday, March 3

Notation: Define

$$\rho(q) = [\mathbb{Z}_q^\times : (\mathbb{Z}_q^\times)^2]$$
$$= \#\{\text{real characters} \pmod q\}$$
$$= \#\{x \pmod q : x^2 \equiv 1 \pmod q\}.$$

If q is odd, then $\rho(q) = 2^{\omega(q)}$ (where $\omega(q)$ is the number of distinct prime factors of q). Always $\rho(q) \le \tau(q) \ll_\varepsilon q^\varepsilon$ (where $\tau(q)$ is the number of divisors of q).

- Define

$$c(q,a) = -1 + \#\{x \pmod q : x^2 \equiv a \pmod q\}.$$

When $(a,q)=1$, $c(q,a)$ can only take the values -1 or $\rho(q)-1$.

We've seen that

$$E^\pi(x;q,a) = \frac{\phi(q)\pi(x;q,a) - \cancel{\mathrm{li}(x)}\ \pi(x)}{\sqrt{x}/\log x}$$
$$= -c(q,a) + [\text{explicit formula}].$$

More notation:

$$E^\psi(x;q,a,b) = \frac{\phi(q)(\psi(x;q,a) - \psi(x;q,b))}{\sqrt{x}}$$
$$= \sum_{\chi \pmod q} (\bar\chi(b) - \bar\chi(a)) \sum_{\substack{\rho \\ L(\rho,\chi)=0 \\ \text{nontrivial}}} \frac{x^{\rho-\frac12}}{\rho} + o(1).$$

and

$$E^\pi(x;q,a,b) = \frac{\phi(q)(\pi(x;q,a) - \pi(x;q,b))}{\sqrt{x}/\log x}$$
$$= \underbrace{c(q,b) - c(q,a)}_{\text{e.g., } = \rho(q) \text{ if } a \ne \square,\ b = \square} + E^\psi(x;q,a,b) + o(1).$$

Let's start assuming GRH.

$$E^{\pi}(x;q,a,b) = c(q,b) - c(q,a) - \sum_{\chi \,(\mathrm{mod}\, q)} (\bar\chi(b) - \bar\chi(a)) \sum_{\substack{\gamma \\ L(\frac12+i\gamma,\chi)=0}} \frac{x^{i\gamma}}{\frac12+i\gamma} + o(1)$$

[Side note: if $f(x) = g(x) + o(1)$, then $f(x)$ and $g(x)$ have the same limiting logarithmic distributions.]

Modulo lots of convergence considerations, we do have — assuming LI, linear independence of $\{\gamma\}$:

$$\hat{E}^{\pi}(t;q,a,b) = e^{it(c(q,b)-c(q,a))} \times \prod_{\chi \,(\mathrm{mod}\, q)} \prod_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} J_0\left(\frac{2|\chi(b)-\chi(a)|}{\sqrt{\frac14+\gamma^2}}\, t\right),$$

This function $\hat{E}^{\pi}$ is also the characteristic function of the random variable

$$X^{\pi}(q;a,b) = c(q,b) - c(q,a) + \sum_{\chi \,(\mathrm{mod}\, q)} |\chi(b)-\chi(a)| \sum_{\substack{\gamma>0 \\ L(\frac12+i\gamma,\chi)=0}} \frac{X_\gamma}{\sqrt{\frac14+\gamma^2}},$$

where each $X_\gamma = 2\,\mathrm{Re}\, Z_\gamma$ with Z_γ uniformly distributed on S^1, and $\{Z_\gamma\}$ independent.

Define $\delta_{q;a,b}$ to be the logarithmic density of the set

$$S_{q;a,b} = \{x \ge 1 : \pi(x;q,a) > \pi(x;q,b)\},$$

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that is, $\displaystyle \lim_{y\to\infty} \frac{1}{\log y} \int_1^y \mathbb{1}_{S_{q;a,b}}(x) \frac{dx}{x}$.

Note $S_{q;a,b} = \{ x \geq 1 : E^{\pi}(x;q,a,b) > 0 \}$.

Since the limiting logarithmic distribution of $E^{\pi}(x;q,a,b)$ is the same as the random variable $X^{\pi}(q;a,b)$, we're looking at

$$\delta_{q;a,b} = \Pr(X^{\pi}(q;a,b) > 0).$$

assuming $\{0\}$ isn't assigned mass; we want distribution continuous

(limit LI implies this)

We can then calculate $\delta_{q;a,b}$ as follows:

Let $\mu_{q;a,b}$ be the limiting distribution shared by $E^{\pi}(x;q,a,b)$ and $X^{\pi}(q;a,b)$.

Then $\displaystyle \delta_{q;a,b} = \int_0^\infty d\mu_{q;a,b}$

$$= \int_{\mathbb{R}} \mathbb{1}_{(0,\infty)}(x)\, d\mu_{q;a,b}(x)$$

By Plancherel (Parseval)

$$\delta_{q;a,b} = \int_{\mathbb{R}} \hat{\mathbb{1}}_{(0,\infty)}(t)\, \hat{\mu}_{q;a,b}(t)\, dt$$

If we use "$\hat{\mathbb{1}}_{(0,\infty)}(t) = \frac{1}{2}\left(\delta(t) - \frac{i}{\pi t} \right)$" $\underbrace{\delta(t)}_{\text{Dirac}}$

then

$$\delta_{q,a,b} = \int_{\mathbb{R}} \frac{1}{2}\left(\delta(t) - \frac{i}{\pi t}\right)\hat{E}^{\pi}(t;q,a,b)\,dt$$

$$= \frac{1}{2}\hat{E}^{\pi}(0;q,a,b) - \frac{i}{2\pi}\int_{\mathbb{R}} \hat{E}^{\pi}(t,q,a,b)\frac{dt}{t}$$

$$= \frac{1}{2} + \frac{1}{2\pi}\int_{\mathbb{R}} \left[\frac{\sin\left(t(c(q,a)-c(q,b))\right)}{t} \times \right.$$

$$\left. \prod_{\chi} \prod_{\gamma} J_0\left(\frac{2|\chi(a)-\chi(b)|}{\sqrt{1/4+\gamma^2}}\,t\right)\right] dt.$$