Friday, March 31

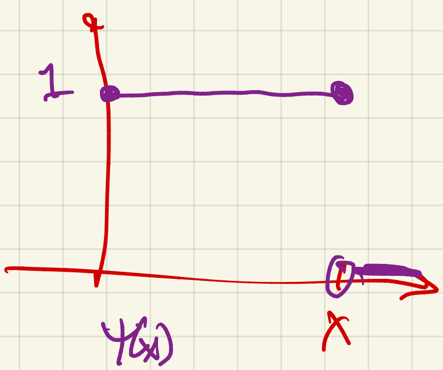
(Material: MV, Section 5.1 ; or Iwaniec – Kowalski)

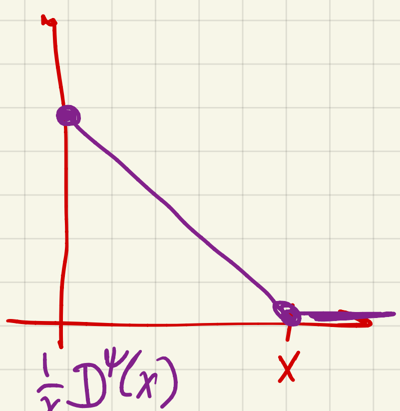
Warm-up calculations

- $$D^{\psi}(x) = \int_0^x \psi(t)\,dt = \int_0^x \sum_{n\leq t} \Lambda(n)\,dt = \sum_{n\leq x} \Lambda(n) \int_n^x dt = \sum_{n\leq x} \Lambda(n)(x-n).$$

Side note: can also write

$$\frac{1}{x}\int_0^x \psi(t)\,dt = \sum_{n\leq x} \Lambda(n)\left(1-\frac{n}{x}\right).$$





- $$D^{\psi}(x) = \int_0^x \left(\frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'}{\zeta}(s)\,\frac{t^s}{s}\,ds\right)dt$$

$$= \frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'}{\zeta}(s)\,\frac{1}{s}\left(\int_0^x t^s\,dt\right)ds$$

$$= \frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'}{\zeta}(s)\,\frac{x^{s+1}}{s(s+1)}\,ds$$

- $$D^{\psi}(x) = \int_0^x \left(t - \sum_{\rho}\frac{t^{\rho}}{\rho} + \text{error}\right)dt$$

$$= \frac{x^2}{2} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} + \text{error}$$

More generally:

- any sequence $\{a_n\}$
- $D_0(x) = \sum_{n\leq x} a_n$
- $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$

Perron's formula: $D_0(x) = \frac{1}{2\pi i}\int_{(c)} \alpha(s)\frac{x^s}{s}\,ds$

Define $D_k(x) = \int_0^x D_{k-1}(t)\,dt$.

We can check:

$$D_k(x) = \sum_{n\le x} a_n \frac{(x-n)^k}{k!}$$

and

$$D_k(x) = \frac{1}{2\pi i}\int_{(c)} \alpha(s)\frac{x^{s+k}}{s(s+1)\cdots(s+k)}\,ds$$

A similar smoothing/averaging scheme:

- $\{a_n\}$, $\alpha(s)$ as before
- $\Delta_0(x) = \sum_{n\le x} a_n$

Now set

$$\Delta_k(x) = \int_0^x \Delta_{k-1}(t)\,\frac{dt}{t}$$

We can check:

$$\Delta_k(x) = \sum_{n\le x} a_n \frac{1}{k!}\left(\log\frac{x}{n}\right)^k$$

and also

$$\Delta_k(x) = \frac{1}{2\pi i}\int_{(c)} \alpha(s)\frac{x^s}{s^{k+1}}\,ds$$

One other scheme:

- $\{a_n\}$, $\alpha(s)$ as before
- $P(x) = \sum_{n<1}^{\infty} a_n e^{-n/x}$.
  - approximating the abrupt cutoff function at $n=x$

It turns out that

$$P(x) = \frac{1}{2\pi i}\int_{(c)} \alpha(s)\,\Gamma(s)\,x^s\,ds.$$

(Γ is the Euler gamma function)

In particular,

$$\Delta_k^{\psi}(x) = \frac{1}{k!} \sum_{n \le x} \Lambda(n) \left(\log \frac{x}{n}\right)^k$$

$$= x - \sum_{\rho} \frac{x^{\rho}}{\rho^{k+1}} + \text{error}$$

On Monday we'll look at

- behaviour / distribution of $-\sum_{\rho} \frac{x^{\rho}}{\rho^{k+1}}$
- relate sign changes of $\Delta_k^{\psi}(x) - x$ to sign changes of $\psi(x) - x$.