Asymptotic formulas for two-way prime number race logarithmic densities

Recall : under GRH,

$$E(x;q,a,b) := \phi(q)\frac{\pi(x;q,a)-\pi(x;q,b)}{\sqrt{x}/\log x}$$

$$= c(q,b)-c(q,a)+\sum_{\chi \bmod q}(\overline{\chi}(b)-\overline{\chi}(a))\sum_{\gamma_\chi}\frac{x^{i\gamma_\chi}}{\frac{1}{2}+i\gamma_\chi}+o(1),$$

$$c(q,a) := \#\{k \bmod q : k^2 \equiv a \bmod q\}.$$

Also, $E(e^y;q,a,b)$ has a limiting distribution $\mu_{q;a,b}$. It is the probability measure associated with a random variable $X_{q,a,b}$.

Two way prime number races

We are interested in $\delta(q; a, b)$, the logarithmic density of the set $\{x > 0 : E(x; q, a, b) > 0\}$, that is the "probability" that $\pi(x; q, a) - \pi(x; q, b)$. Under GRH and LI, we have

$$\delta(q; a, b) = \mathbb{P}[X_{q;a,b} > 0].$$

Our goal is to find an asymptotic formula for $\delta(q; a, b)$ as $q \to \infty$. This was done by F. and Martin.

Under GRH+LI,

$$X_{q;a,b} \stackrel{d}{=} c(q,b) - c(q,a) + \sum_{\chi \bmod q} |\overline{\chi}(b) - \overline{\chi}(a)| \sum_{\gamma_\chi > 0} \frac{X_{\gamma_\chi}}{\sqrt{\frac{1}{4} + \gamma_\chi^2}},$$

where $X_{\gamma_\chi} = \Re e^{2\pi i Y}$ are i.i.d., $Y \sim U[0,1]$.

$$\mathbb{E}[X_{q;a,b}] = c(q,b) - c(q,a);$$

$$\mathbb{V}[X_{q;a,b}] = \sum_{\chi \bmod q} |\overline{\chi}(b) - \overline{\chi}(a)|^2 \sum_{\gamma_\chi} \frac{1}{\frac{1}{4} + \gamma_\chi^2}$$

Note : $\mathbb{E}[X_{q;a,b}] = q^{o(1)}$, $\mathbb{V}[X_{q;a,b}] = q^{1-o(1)}$.

Recall : under GRH+LI, the logarithmic density $\delta(q;a,b)$ of the set $\{x>0 : E(x;q,a)>0\}$ exists and equals $\mathbb{P}(X_{q;a,b}>0)$. By independence of the X_{γ_χ},

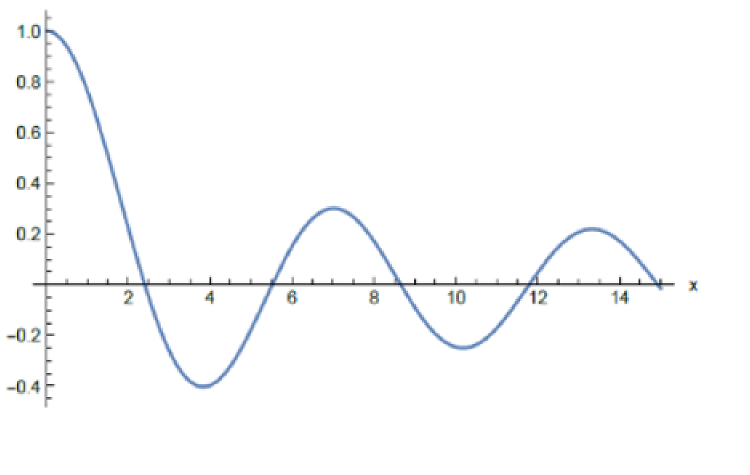
$$\widehat{X}_{q;a,b}(\xi) := \mathbb{E}[e^{i\xi X_{q;a,b}}] = \mathbb{E}[e^{i\xi(c(q,b)-c(q,a))}]\mathbb{E}[e^{i\sum_{\chi,\gamma_\chi} \frac{|\overline{\chi}(b)-\overline{\chi}(a)|X_{\gamma_\chi}}{\sqrt{\frac14+\gamma_\chi^2}}}]$$

$$= e^{i\xi\mathbb{E}[X_{q;a,b}]}\prod_{\chi \bmod q}\prod_{\gamma_\chi>0}\mathbb{E}[e^{i\xi|\overline{\chi}(b)-\overline{\chi}(a)|\frac{X_{\gamma_\chi}}{\sqrt{\frac14+\gamma_\chi^2}}}]$$

$$= e^{i\xi\mathbb{E}[X_{q;a,b}]}\prod_{\chi \bmod q}\prod_{\gamma_\chi>0} J_0\Big(\frac{2\xi|\overline{\chi}(b)-\overline{\chi}(a)|}{\sqrt{\frac14+\gamma_\chi^2}}\Big).$$

The last step follows from the identity $J_0(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\cos t}dt$.

The Bessel function



How to recover the CDF (at 0) of $X_{q;a,b}$ from this formula ? One can do it via Berry-Esseen. Let $\Phi(t)$ be the CDF of the Gaussian.

Theorem (Berry-Esseen)

Let Y be a real-valued random variable. For $T > 0$,

$$\sup_{t\in\mathbb{R}} |F_Y(t) - \Phi(t)| \ll \int_{|\xi|<T} \frac{|\widehat{Y}(\xi) - e^{-\frac{\xi^2}{2}}|}{\xi} d\xi + \frac{1}{T}.$$

We will pick $Y = (X_{q;a,b} - \mathbb{E}[X_{q;a,b}])/\mathbb{V}[X_{q;a,b}]^{\frac{1}{2}}$. Note that

$$\delta(q;a,b) = \mathbb{P}[Y > -\mathbb{E}[X_{q;a,b}]/\mathbb{V}[X_{q;a,b}]^{\frac{1}{2}}].$$

Estimating the characteristic function

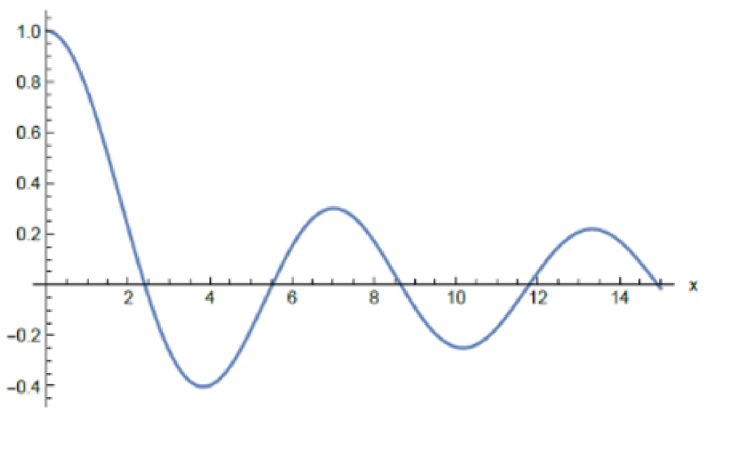
To use Berry-Essen, we need bounds on the characteristic function $\widehat{Y}(\xi) = e^{-i\xi\mathbb{E}[X_{q;a,b}]}\widehat{X}(\xi/V^{\frac{1}{2}})$, where $V := \mathbb{V}[X_{q;a,b}] = q^{1-o(1)}$.

Let $|\xi| > 200V^{\frac{1}{2}}$. From the bounds $|J_0(u)| \leq 1, |J_0(u)| \ll |u|^{-\frac{1}{2}}$, we can pick out a positive proportion of characters for which $|\chi(a) - \chi(b)| \geq \frac{1}{4}$ and the zeros for which $|\gamma_\chi| \leq |\xi|/2V^{\frac{1}{2}}$ to deduce that

$$\widehat{Y}(\xi) \ll \prod_{\chi \in S_q} \prod_{0<\gamma_\chi<|\xi|/2} \frac{(\frac{1}{4} + \gamma_\chi^2)^{\frac{1}{4}}}{(|\xi|/V^{\frac{1}{2}})^{\frac{1}{2}}} \ll e^{-c\phi(q)|\xi|/V^{\frac{1}{2}}},$$

where S_q is some subset of characters of positive proportion. This is exponentially small in q.

The Bessel function



Estimating the characteristic function

An exponentially small bound also holds for $V^{\frac{1}{2}}/200 < |\xi| \le 200V^{\frac{1}{2}}$. Indeed, if κ is fixed and small enough ($\kappa = \frac{5}{24}$ will do), $|J_0(x)| \le |J_0(\kappa)|$ for all $x > 0$, so

$$\widehat{Y}(\xi) = \prod_{\chi \bmod q} \prod_{\gamma_\chi > 0} J_0\Big(\frac{2\xi|\overline{\chi}(b) - \overline{\chi}(a)|}{\sqrt{V}\sqrt{\frac{1}{4} + \gamma_\chi^2}}\Big)$$
$$\ll \prod_{\chi \bmod q} \prod_{\gamma_\chi > 0} \Big|J_0\Big(\frac{|\overline{\chi}(b) - \overline{\chi}(a)|}{100\sqrt{\frac{1}{4} + \gamma_\chi^2}}\Big)\Big|,$$

which is exponentially small in q by the same arguments as before.

Taylor series

Fact : each coefficient of the Taylor series $\log J_0(u) = -\frac{u^2}{2} - \frac{u^4}{64} + ...$ is negative.

Hence, for $|\xi| < V^{\frac{1}{2}}/200$,

$$\log \widehat{Y}(\xi) = \sum_{\chi \bmod q} \sum_{\gamma_\chi > 0} \log J_0\Big(\frac{2\xi|\overline{\chi}(b) - \overline{\chi}(a)|}{V^{\frac{1}{2}} \sqrt{\frac{1}{4} + \gamma_\chi^2}} \Big)$$

$$= -\frac{\xi^2}{V} \sum_{\chi \bmod q} \sum_{\gamma_\chi > 0} \frac{|\overline{\chi}(b) - \overline{\chi}(a)|^2}{\frac{1}{4} + \gamma_\chi^2} + O\Big(\frac{\xi^4 \phi(q)}{V^2} \Big)$$

$$= -\frac{\xi^2}{2} + O\Big(\frac{\xi^4 \phi(q)}{V^2} \Big).$$

Moreover, in the same range, $\log \widehat{Y}(\xi) \leq -\frac{\xi^2}{2}$, so $|\widehat{Y}(\xi)| \ll e^{-\xi^2/2}$.

Applying Berry-Esseen

Combining these bounds, we have proved :

$$|\widehat{Y}(\xi)| \ll e^{-cq^{\frac{1}{3}}} \qquad (|\xi| > V^{\frac{1}{4}});$$

$$\widehat{Y}(\xi) = e^{-\frac{\xi^2}{2}} e^{O(\frac{\xi^4\phi(q)}{V^2})} = e^{-\frac{\xi^2}{2}}\Big(1 + O\Big(\frac{\xi^4\phi(q)}{V^2}\Big)\Big) \qquad (|\xi| \leq V^{\frac{1}{4}}).$$

Plugging this into Berry-Esseen :

$$\sup_{t\in\mathbb{R}} |F_Y(t) - \Phi(t)| \ll \int_{|\xi|<q^2} \frac{|\widehat{Y}(\xi) - e^{-\frac{\xi^2}{2}}|}{\xi} d\xi + \frac{1}{q^2}$$

$$\ll \int_{|\xi|<V^{\frac{1}{4}}} \frac{|\widehat{Y}(\xi) - e^{-\frac{\xi^2}{2}}|}{\xi} d\xi + \int_{V^{\frac{1}{4}}<|\xi|<q^2} e^{-cq^{\frac{1}{3}}} d\xi + \frac{1}{q^2}.$$

Applying Berry-Esseen

By the Taylor series expansion, the first integral is

$$\ll \int_{|\xi|<q^{\varepsilon}} e^{-\frac{\xi^2}{2}} \frac{\xi^4 \phi(q)}{V^2 \xi} d\xi \ll_{\varepsilon} \frac{1}{q^{1-\varepsilon}}.$$

Finally, what we are really looking for is

$$\mathbb{P}[X_{q;a,b} > 0] = \mathbb{P}[Y > -\mathbb{E}[X_{q;a,b}]/V^{\frac{1}{2}}] = F_Y(-\mathbb{E}[X_{q;a,b}]/V^{\frac{1}{2}}),$$

which by Berry-Esseen is equal to $\Phi(-\mathbb{E}[X_{q;a,b}]/V^{\frac{1}{2}}) + O_{\varepsilon}(1/q^{1-\varepsilon})$. However, applying Taylor series,

$$\frac{1}{\sqrt{2\pi}} \int_{-\mathbb{E}[X_{q;a,b}]/V^{\frac{1}{2}}}^{\infty} e^{-\frac{u^2}{2}} du = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{-\mathbb{E}[X_{q;a,b}]/V^{\frac{1}{2}}}^{0} e^{-\frac{u^2}{2}} du$$
$$= \frac{1}{2} + \frac{\mathbb{E}[X_{q;a,b}]}{(2\pi V)^{\frac{1}{2}}} + O_{\varepsilon}(1/V^{1-\varepsilon}).$$

More precise formulas

To summarize, we have shown that

$$\delta(q;a,b) = \frac{1}{2} + \frac{\mathbb{E}[X_{q;a,b}]}{(2\pi V)^{\frac{1}{2}}} + O_\varepsilon\Big(\frac{1}{V^{1-\varepsilon}}\Big).$$

The error term is actually $O_\varepsilon(V^{-\frac{3}{2}-\varepsilon})$ We can do even better (F. and Martin) : for any fixed K,

$$\delta(q;a,b) = \frac{1}{2} + \frac{\mathbb{E}[X_{q;a,b}]}{(2\pi V)^{\frac{1}{2}}} + \cdots + O_K\Big(\frac{1}{V^{K}}\Big).$$

This can be done via (Rubinstein-Sarnak, Feueuverger-Martin)

$$\delta(q;a,b) = \frac{1}{2} + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sin(\xi V^{-\frac{1}{2}}(c(q,a) - c(q,b)))}{\xi} \times \prod_{\chi \bmod q}\prod_{\gamma_\chi>0} J_0\Big(\frac{2\xi|\overline{\chi}(b) - \overline{\chi}(a)|}{V^{\frac{1}{2}}\sqrt{\frac{1}{4}+\gamma_\chi^2}}\Big) d\xi.$$

More precise formulas

From our earlier bounds, the part of the integral with $|t| > V^{\frac{1}{2}}/200$ is exponentially small. The same is true for the integral in the range $V^{\frac{1}{4}} < |t| < V^{\frac{1}{2}}/200$, since in this range $\widehat{Y}(\xi) \ll e^{-\xi^2/2}$.

In the range $|t| < V^{\frac{1}{4}}$, we can apply Taylor series, both for $\sin(t)/t$ and for the product of Bessel functions.

Cumulants

The cumulant generating function :

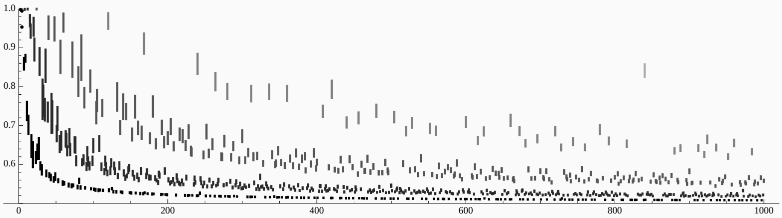
$$\log \widehat{Y}(\xi) = \sum_{\chi \bmod q} \sum_{\gamma_\chi > 0} \log J_0\Big(\frac{2\xi|\overline{\chi}(b) - \overline{\chi}(a)|}{V^{\frac{1}{2}} \sqrt{\frac{1}{4} + \gamma_\chi^2}} \Big)$$

$$= -\frac{\xi^2}{2} + \sum_{\ell=4}^{\infty} \frac{\alpha_\ell \xi^\ell}{V^{\ell/2}} \sum_{\chi \bmod q} \sum_{\gamma_\chi > 0} \frac{|\chi(a) - \chi(b)|^\ell}{(\frac{1}{4} + \gamma_\chi^2)^{\ell/2}}.$$

It turns out that the double sum is $= V^{1+o(1)}$. This can be multiplied by that of $\sin(t)/t$ and integrated, giving an asymptotic series for $\delta(q; a, b)$.

All densities for $q \leq 1000$

Here is a plot of all values of $\delta(q; a, b)$ with $q \leq 1000$. Notice the square-root decay.



Note : $c(q,a) - c(q,b) = 2^k$ with $k = \omega(q) - 1, \omega(q)$ or $\omega(q) + 1$. Those are the bands in the graph.

Thank you !