Wednesday, March 8

Assume GRH and LI.

Recall notation:

- $p(q) = \#\{x \pmod q : x^2 \equiv 1 \pmod q\}$

  $(p(q) \ll_\varepsilon q^\varepsilon)$

- $$V(q;a,b) = \sum_{\chi \pmod q} |\chi(b) - \chi(a)|^2 b(\chi),$$

where $$b(\chi) = \sum_{\substack{\gamma \in \mathbb{R} \\ L(\frac12 + i\gamma, \chi) = 0}} \frac{1}{\frac14 + \gamma^2}.$$

- $$X_{q;a,b} = c(q,b) - c(q,a) + \sum_{\chi \pmod q} |\chi(b) - \chi(a)| \sum_\gamma \frac{Z_\gamma}{\sqrt{\frac14 + \gamma^2}},$$

where Z_γ are independent, unif. dist'd on S^1. Daniel calculated that

$$\mathbb{E}(X_{q;a,b}) = c(q,b) - c(q,a) = \begin{cases} p(q), & \text{if } a \ne \square \text{ and } b = \square, \quad (*) \\ -p(q), & \text{if } a = \square \text{ and } b \ne \square, \\ 0, & \text{otherwise.} \end{cases}$$

$$\sigma^2(X_{q;a,b}) = V(q;a,b).$$

We know $X_{q;a,b}$ is the limiting logarithmic distribution of

$$E(x;q,a,b) = \phi(q) \frac{\pi(x;q,a) - \pi(x;q,b)}{\sqrt{x}/\log x}.$$

In particular,

$$\delta_{q;a,b} = \text{log'c density of } \{x : E(x;q,a,b) > 0\}$$
$$= \Pr(X_{q;a,b} > 0)$$

In the case $(*)$,

$$\delta_{q;a,b} = \frac12 + \Pr\left(0 < N(0,1) < \frac{p(q)}{\sqrt{V(q;a,b)}}\right) + O\left(\frac{1}{V(q;a,b)}\right).$$

(If $a = \square$ and $b \neq \square$, then $\delta_{q;a,b} = \frac{1}{2} - \ldots$ instead of $\frac{1}{2} + \ldots$

If a,b are both squares or both nonsquares, Rubinstein/Sarnak showed $\delta_{q;a,b} = 1/2$.)

Goal: asymptotic formula for $V(q;a,b)$.

$$V(q;a,b) = \sum_{\chi \,(\mathrm{mod}\, q)}' |\chi(b) - \chi(a)|^2 \, b(\chi).$$

- We know $b(\chi) = \log q^* + O(\log\log q)$.

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$$\left(b(\chi) = \sum_{\gamma} \frac{1}{1/4 + \gamma^2}\right).$$

Lemma 3.2 ("Inequities"):

For $q \in \mathbb{N}$ and any $s \mid q$, $1 \le s < q$:

- $$\sum_{d \mid q} \Lambda\left(\frac{q}{d}\right) \phi(d) = \phi(q) \sum_{p \mid q} \frac{\log p}{p-1}$$

- $$\sum_{d \mid s} \Lambda\left(\frac{q}{d}\right) \phi(d) = \phi(q) \frac{\Lambda(q/s)}{\phi(q/s)}.$$

Method of proof: q/d must be a prime power; so group the q/d according to the prime $p \mid q$ such that $q/d = p^r$.

Proposition 3.3: For $q \in \mathbb{N}$, and any $(a,q) = 1$, $a \not\equiv 1 \pmod{q}$:

- $$\sum_{\chi \,(\mathrm{mod}\, q)}' \log q^* = \phi(q)\left(\log q - \sum_{p \mid q} \frac{\log p}{p-1}\right)$$

- $$\sum_{\chi \,(\mathrm{mod}\, q)}' \chi(a) \log q^* = -\phi(q) \frac{\Lambda(q/(q,a-1))}{\phi(q/(q,a-1))}.$$

Proof: (At first, allow $a \equiv 1 \pmod q$ as well.)

We start by evaluating

$$\sum_{d \mid q} \Lambda(q/d) \sum_{\chi \,(\mathrm{mod}\, d)} \chi(a)$$

$$= \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \sum_{\substack{d \mid q \\ q^* \mid d}} \Lambda(q/d)$$

$(c = q/d)$

$$= \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \sum_{c \mid q/q^*} \Lambda(c)$$

$$= \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \log(q/q^*)$$

$$= \log q \cdot \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) - \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \log q^*.$$

- If $a \equiv 1 \pmod q$ then we've shown

$$\sum_{d \mid q} \Lambda(q/d)\, \phi(d) = \log q \cdot \phi(q) - \sum_{\chi \,(\mathrm{mod}\, q)} \log q^*,$$

and so we're done by Lemma 3.2.

Now assume $a \not\equiv 1 \pmod q$. Then

$$\sum_{\chi \,(\mathrm{mod}\, q)} \chi(a) \log q^* = \log q \cdot \cancel{\sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)}^{\,0} - \sum_{d \mid q} \Lambda(q/d) \sum_{\chi \,(\mathrm{mod}\, d)} \chi(a)$$

$$= -\sum_{d \mid q} \Lambda(q/d) \begin{cases} \phi(d), & \text{if } a \equiv 1 \pmod d, \\ 0, & \text{otherwise.} \end{cases}$$

$$= -\sum_{\substack{d \mid q \\ d \mid (a-1)}} \Lambda(q/d)\, \phi(d)$$

$(s = \gcd(q, a-1))$

$$= -\phi(q) \frac{\Lambda(q/(q,a-1))}{\phi(q/(q,a-1))} \text{ by Lemma 3.2. } //$$

Now consider

$$V(q;a,b) = \sum_{\chi \,(\mathrm{mod}\, q)} |\chi(b)-\chi(a)|^2 \, b(\chi)$$

$$= \sum_{\chi\,(\mathrm{mod}\, q)} (\chi(b)-\chi(a))(\bar\chi(b)-\bar\chi(a)) \times \left(\log q^* + O(\log\log q)\right)$$

$$= \sum_{\chi\,(\mathrm{mod}\, q)} (2-\chi(ab^{-1})-\chi(ba^{-1}))\log q^* + O(\phi(q)\log\log q).$$

From Proposition 3.3,

$$= 2\phi(q)\Big(\log q - \sum_{p\mid q}\frac{\log p}{p-1}\Big) + \phi(q)\frac{\Lambda(q/(q,ab^{-1}-1))}{\phi(q/(q,ab^{-1}-1))}$$

$$+ (\text{same with } ba^{-1} \text{ instead of } ab^{-1}) + O(),$$

But since $(b,q)=1$,

$$(q, ab^{-1}-1) = (q, b(ab^{-1}-1)) = (q, a-b)$$

and similarly $(q, ba^{-1}-1) = (q, b-a) = (q, a-b)$.

Hence

$$V(q;a,b) = 2\phi(q)\Big(\log q - \sum_{p\mid q}\frac{\log p}{p-1} + \frac{\Lambda(q/(q,a-b))}{\phi(q/(q,a-b))} + O(\log\log q)\Big).$$

Note $\frac{\Lambda(n)}{\phi(n)} \ll \frac{\log n}{n^{1-\varepsilon}}$ is uniformly bounded.

Since $\frac{\log t}{t-1}$ is decreasing,

$$\sum_{p\mid q}\frac{\log p}{p-1} \le \sum_{j=1}^{\omega(q)}\frac{\log p_j}{p_j-1} \ll \log\log q.$$

$$\ll \sum_{p<\omega(q)\log\omega(q)}\frac{\log p}{p} \ll \log(\omega(q)\log\omega(q)) \cancel{\ll \log q}. \quad \text{since } \omega(q) \ll \frac{\log q}{\log_2 q}.$$

We've shown

$$V(q;a,b) = 2\phi(q)\log q + O(\phi(q)\log\log q)$$

$$= 2\phi(q)\log q \cdot \left(1 + O\left(\frac{\log\log q}{\log q}\right)\right).$$

Therefore (if $a \equiv \square$, $b \not\equiv \square$)

$$\delta_{q;a,b} = \frac{1}{2} + \Pr\left(0 < N(0,1) < \frac{\rho(q)}{\sqrt{V(q;a,b)}}\right) + O\left(\frac{1}{V(q;a,b)}\right)$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\rho(q)}{\sqrt{V(q;a,b)}} + O(-)$$

$$= \frac{1}{2} + \frac{\rho(q)}{\sqrt{2\pi}}\frac{1}{\sqrt{2\phi(q)\log q}}\left(1 + O\left(\frac{\log\log q}{\log q}\right)\right) + O(-)$$

$$= \frac{1}{2} + \frac{\rho(q)}{2\sqrt{\pi}\cdot\sqrt{\phi(q)\log\log q}}\left(1 + O\left(\frac{\log\log q}{\log q}\right)\right)$$