THE PROBLEMS OF PÓLYA AND TURÁN

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ABSTRACT. The topic of this paper is the problems of Pólya and Turán. We will introduce Pólya's problem and Turán's problem. Then we proceed to discuss about connection between of $L_\alpha(x) = \sum_{n\leq x} \frac{\lambda(n)}{n^\alpha}$ to the Riemann hypothesis. Finally, we will illustrate some special cases of $L_\alpha(x)$ and the conclusion.

1. PRELIMINARIES

First, let's recall the definition of the Liouville function and the sum of the Liouville function; also, the lemma of Perron's formula.

Definition 1.1. Let $\lambda(n)$ denote the Liouville function, defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^k$ if n is composed of k not necessarily distinct prime factors (i.e., if $n = \prod_{i=1}^k p_i^{\alpha_i}$ then $\lambda(n) = \prod_{i=1}^k (-1)p_i^{\alpha_i}$).

Notation 1.2. The Liouville function $\lambda(n)$ is completely multiplicative.

Definition 1.3. [3] The Dirichlet series for the Liouville function is related to the Riemann zeta function by

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} = \prod_p (1 + \frac{1}{p^s})^{-1} \tag{1.1}$$

if $\sigma > 1$, where $s = \sigma + it$.

Definition 1.4. The sum of the Liouville function $L(x)$, defined by $L(x) = \sum_{n\leq x} \lambda(n)$.

Lemma 1.5. [3] *Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ $(\sigma > 1)$, where $a_n = O(\psi(n))$, $\psi(n)$ being non-decreasing, and*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = O(\frac{1}{(\sigma-1)^\alpha})$$

as $\sigma \to 1$. Then if $c > 0, \sigma + c > 1, x$ is not an integer, and N is the integer nearest to x,

$$\sum_{n\leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + O(\frac{x^c}{T(\sigma+c-1)^\alpha}) + O(\frac{\psi(2x)x^{1-\sigma}\log x}{T}) + O(\frac{\psi(N)x^{1-\sigma}}{T|x-N|}).$$

If x is an integer, the corresponding result is

$$\sum_{n=1}^{x-1} \frac{a_n}{n^s} + \frac{a_x}{2x^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} dw + O(\frac{x^c}{T(\sigma+c-1)^\alpha}) + O(\frac{\psi(2x)x^{1-\sigma}\log x}{T}) + O(\frac{\psi(N)x^{-\sigma}}{T}).$$

2. Pólya's problem

Pólya's problem, named after the Hungarian mathematician George Pólya in 1919[4], concerns the behaviour of the sum of the Liouville function. He observed that

$$L(x) = \sum_{n \leq x} \lambda(n) \leq 0 \text{ (for } x \geq 2). \tag{2.1}$$

Also, $L(x)$ tells us the difference between the number of prime factors with an even number and those with an odd number of elements up to x. According to Pólya, the Riemann hypothesis would follow if $L(x)$ eventually has the constant sign.

Problem 2.1 (Pólya's problem)**.** Show that $L(x)$ changes sign infinitely often, and determine the smallest $x \geq 2$ where $L(x) > 0$.

Notation 2.2. Figure 2.1 is a graph comparing $L(x)$ and $y = 0$ for $x \leq 1,000,000$ to make people believe Pólya's problem is true.

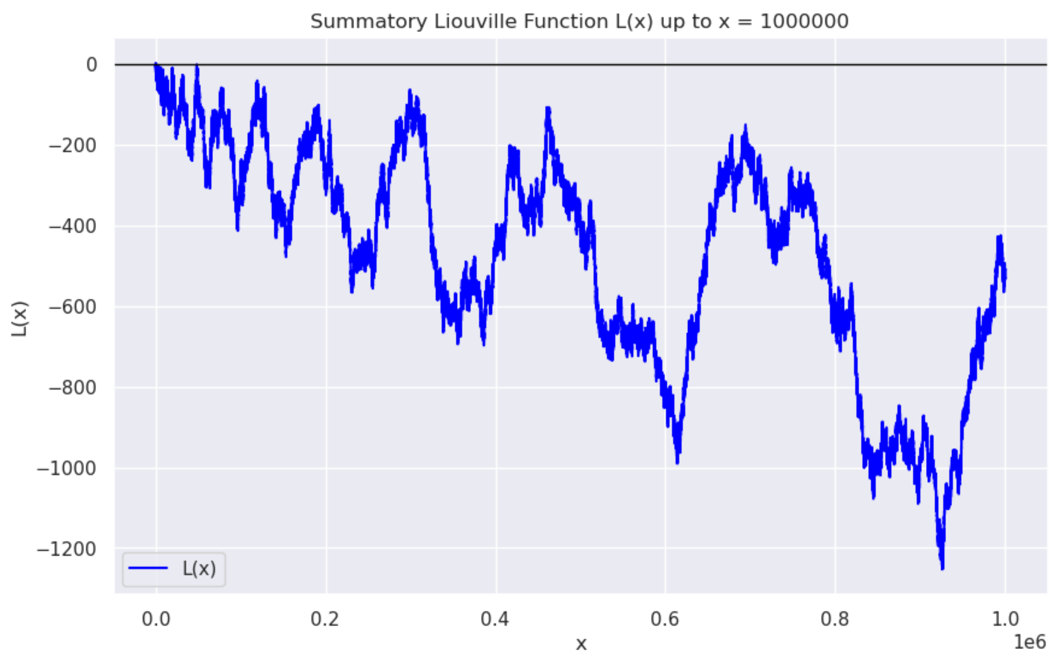

FIGURE 2.1. The function L(x).

Pólya's problem remained open for many years until it was disproved by the British mathematician Alan Haselgrove in 1958[2], which was a surprising result. He constructed a counter-example to the problem based on Ingham's results in 1942[1]. With the Riemann hypotheses and the simplicity of the zeroes, he defined two function

$$A(u) = L(e^u)e^{-\frac{u}{2}} \tag{2.2}$$

and

$$A_T^*(u) = \alpha_0 + 2\Re \sum_{0<\gamma_n<T} (1 - \frac{|\gamma_n|}{T})\alpha_n e^{i\gamma_n u}, \tag{2.3}$$

where $n = 1, 2, \cdots$, and $T > 0, \alpha_0 = 1/\zeta(1/2)$, $\gamma_0 = 0$ and γ_n runs through the imaginary parts of the zeros $\rho_n = 1/2 + i\gamma_n$ of ζ, and $\alpha_n = \zeta(2\rho_n)/\rho_n\zeta'(\rho_n)$.

Theorem 2.3. [1] *Let*

$$F(s) = \int_0^\infty A(u)e^{-su}du,$$

where $A(u)$ is absolutely integrable over every finite interval $0 \le u \le U$, and the integral is convergent in some half-plane $\sigma > \sigma_1 \ge 0$.
Let $A^*(u)$ be a real trigonometrical polynomial

$$A^*(u) = \sum_{n=-N}^{N} \alpha_n e^{i\gamma_n u} \quad (\gamma_n \textit{ real},\ \gamma_{-n} = -\gamma_n,\ \alpha_{-n} = \bar{\alpha}_n)$$

and let

$$F^*(s) = \int_0^\infty A^*(u)e^{-su}du = \sum_{-N}^{N} \frac{\alpha_n}{s - i\gamma_n} \quad (\sigma > 0)$$

Suppose that $F(s) - F^*(s)$ (suitably defined outside the half-plane $\sigma > \sigma_1$) is regular in the region $\sigma \ge 0, -T \le t \le T$, for some $T > 0$ (or, more generally, continuous in this region and regular in the interior).

Then, when $u \to \infty$ (T fixed)

$$\begin{aligned} \liminf A(u) &\le \liminf A_T^*(u) \\ \limsup A_T^*(u) &\le \limsup A(u) \end{aligned} \tag{2.4}$$

where

$$A_T^*(u) = \sum_{|\gamma_n|<T} (1 - \frac{|\gamma_n|}{T})\alpha_n e^{i\gamma_n u} = \alpha_0 + 2\Re \sum_{0<\gamma_n<T} (1 - \frac{|\gamma_n|}{T})\alpha_n e^{i\gamma_n u}$$

Proof. See, e.g., [1], Theorem 1, p.315-316. □

Based on theorem 2.3, if we can find T, u such that $A_T^*(u) > 0$, it will follow that $\limsup A_T^*(u) > 0$, it will follow that $\limsup A(u)$ from equation (2.4). Hence, that $A(u) > 0$, it will follow that $L(e^u)e^{-\frac{u}{2}} > 0 \Rightarrow L(e^u) > 0$ for some u, i.e. that Pólya's problem is false. If the Riemann hypothesis fails, Pólya's problem would be false, so it does not matter whether Ingham bases his argument on Riemann's assumption. With the help of electronic computers, Haselgrove determined $A_T^*(u)$ is positive. Then he got the table 2.1 of $A_T^*(u)$ when $T = 1000$ and $u = 831.800$ to $u = 831.859$. This result would lead us to suspect that $L(e^u)$ becomes positive in the neighbourhood of $u = 831.847$. Therefore, Pólya's problem is false.

u	$A_T^*(u)$	u	$A_T^*(u)$	u	$A_T^*(u)$	u	$A_T^*(u)$
831.800	−0.43329	831.801	−0.42140	831.802	−0.41040	831.803	−0.40181
831.804	−0.439640	831.805	−0.39382	831.806	−0.39287	831.807	−0.39220
831.808	−0.39097	831.809	−0.38918	831.810	−0.38762	831.811	−0.38723
831.812	−0.38853	831.813	−0.39107	831.814	−0.39325	831.815	−0.39265
831.816	−0.38674	831.817	−0.37380	831.818	−0.35378	831.819	−0.32850
831.820	−0.30119	831.821	−0.27534	831.822	−0.25347	831.823	−0.23640
831.824	−0.22333	831.825	−0.21269	831.826	−0.20305	831.827	−0.19370
831.828	−0.18445	831.829	−0.17512	831.830	−0.16518	831.831	−0.15397
831.832	−0.14152	831.833	−0.12920	831.834	−0.11960	831.835	−0.11547
831.836	−0.11807	831.837	−0.12600	831.838	−0.13514	831.839	−0.13999
831.840	−0.13583	831.841	−0.12063	831.842	−0.09590	831.843	−0.06610
831.844	−0.03705	831.845	−0.01395	831.846	0.00014	831.847	0.00495
831.848	0.00265	831.849	−0.00328	831.850	−0.00950	831.851	−0.01404
831.852	−0.01693	831.853	−0.01981	831.854	−0.02493	831.855	−0.03390
831.856	−0.04698	831.857	−1.56321	831.858	−0.08124	831.859	−0.10024

TABLE 2.1. Table of $A_T^*(u)$ when $T = 1000$[2]

In 1960, Sherman Lehman[9] found the smaller value of u when $A_T^\star(u)$ is positive is 79.28 ($T = 1000$) using an IBM 701 at the University of California. He used numerical computation to find the first explicit counter-example and saw $L(906, 180, 359) = 1$. It was 20 years later that Minoru Tanaka[6] concluded Pólya's problem was true when $2 \leq x \leq 2,906,150,256$. Accordingly, $L(x) > 0$ for $x = 906,150,257$ is the smallest counter-example (shown in figure 2.2).

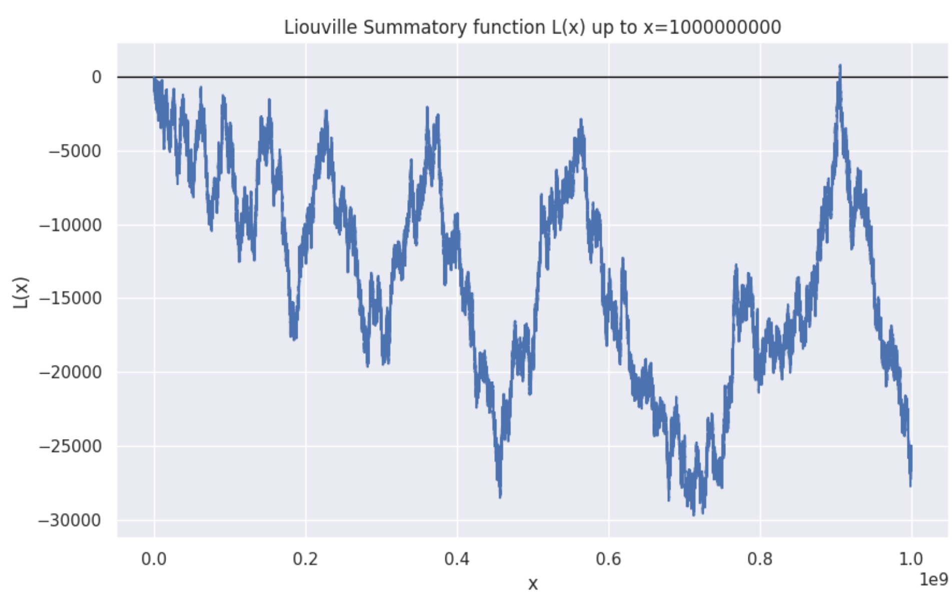

FIGURE 2.2. Counter example of Pólya's problem.

3. Turán's problem

Turán's problem, named after the Hungarian mathematician Paul Turán in 1948[8], concerns some properties of a weighted sum involving the Liouville function. He observed that

$$T(x) = \sum_{n \le x} \frac{\lambda(n)}{n} \ge 0 \text{ (for } x \ge 1). \tag{3.1}$$

Notation 3.1. Figure 3.1 is a graph comparing $T(x)$ and $y = 0$ for $x \le 1,000,000$ to make people believe Turán's problem is true.

Problem 3.2 (Turán's problem)**.** Show that $T(x)$ changes sign infinitely often, and determine the smallest $x \ge 1$ where $L(x) < 0$.

Alan Haselgrove in 1958[2] also used the result of Ingham[1] to show $T(x)$ change sign infinitely often, same as $L(x)$. In 2008, Borwein, Ferguson and Mossinghoff determined that the smallest x where $T(x) < 0$ is $x = 72,186,376,951,205$ by the theorem 3.3 from them.

Theorem 3.3. [7] *Let $T(x)$ denote them sum $\sum_{n=1}^{x} \lambda(n)/n$. The smallest positive integer x for which $T(x) < 0$ is $x = 72,186,376,951,205$, and the minimal value of $T(x)$ for $x \le 7.5 \times 10^{13}$ is $T(72,186,376,951,205) \approx -2.0757641 \times 10^{-9}$.*

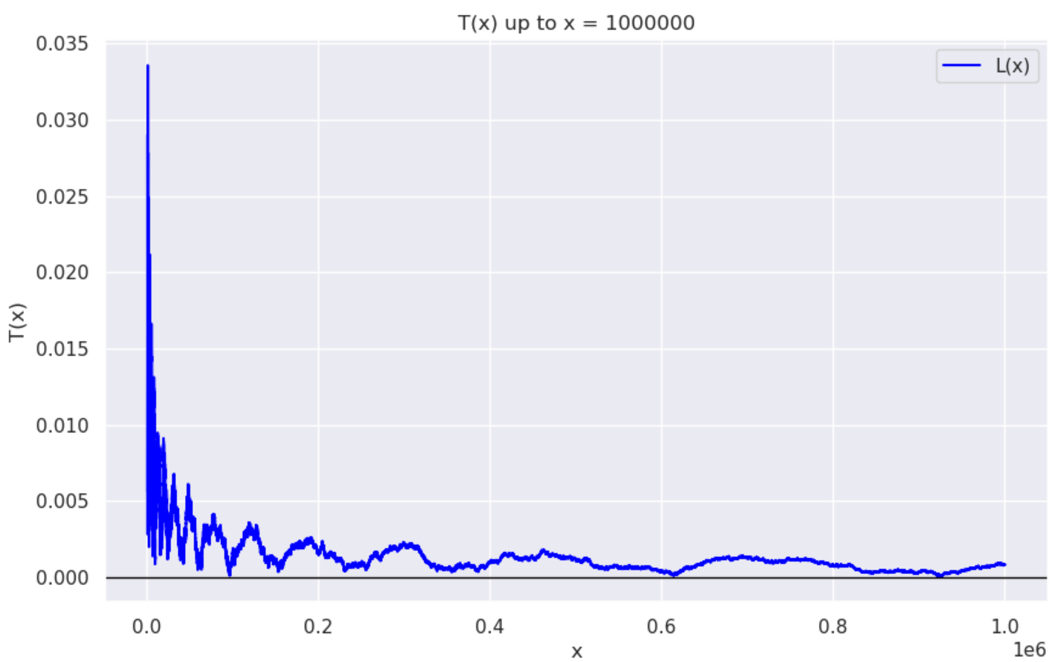

FIGURE 3.1. The function T(x).

4. The connection between $L_\alpha(x)$ and the Riemann hypothesis

In this section, we present some lemma and theorem that reveal the profound relationship between $L_\alpha(x)$ (defined below) and the Riemann hypothesis. For $\alpha \ge 0$, we defines:

$$L_\alpha(x) = \sum_{n \le x} \frac{\lambda(n)}{n^\alpha}, \tag{4.1}$$

so that $L_0(x) = L(x)$ and $L_1(x) = T(x)$.

Using the Euler product, we have the following:

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\alpha} = \prod_p \frac{(1-\frac{1}{p^\alpha})}{(1-\frac{1}{p^{2\alpha}})} = \prod_p (1+\frac{1}{p^\alpha})^{-1} \tag{4.2}$$

If $\alpha > 1$, then $L_\alpha(x)$ converges absolutely to $\zeta(2\alpha)/\zeta(\alpha) > 0$. As a result, this situation can only result in a finite number of sign changes. Hence, we just consider for case where $0 \leq \alpha \leq 1$.

Theorem 4.1. [5] *If the Riemann hypothesis hold then for all* $\alpha \in (1/2, 1]$ *and all* $\varepsilon > 0$

$$L_\alpha(x) = \frac{\zeta(2\alpha)}{\zeta(\alpha)} + O(x^{\frac{1}{2}-\alpha+\varepsilon})$$

As well as conversely if

$$\lim_{x\to\infty} L_\alpha(x) = \frac{\zeta(2\alpha)}{\zeta(\alpha)}$$

for all $\alpha \in (1/2, 1]$*, then the Riemann hypothesis is true.*

Proof. Using lemma 1.5 on the function

$$f(s) = \sum_{n\geq 1} \frac{\lambda(n)}{n^{\alpha+s}} = \frac{\zeta(2(\alpha+s))}{\zeta(\alpha+s)}$$

for $\sigma > 1-\alpha$, we obtain some valuable information on $L_\alpha(x)$. It follows that for $\alpha \in [0, 1]$,

$$L_\alpha(x) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{\zeta(2(\alpha+z))x^z}{\zeta(\alpha+z)z} dz + O\left(x^2/T\right)$$

This leads us to the following:

$$L_\alpha(x) = \frac{1}{2\pi i}\left(\int_{2-iT}^{\frac{1}{2}-\alpha+\delta-iT} + \int_{\frac{1}{2}-\alpha+\delta-iT}^{\frac{1}{2}-\alpha+\delta+iT} + \int_{\frac{1}{2}-\alpha+\delta+iT}^{2+iT}\right)\frac{\zeta(2(\alpha+z))x^z}{\zeta(\alpha+z)z} dz + \frac{\zeta(2\alpha)}{\zeta(\alpha)} + O\left(\frac{x^2}{T}\right)$$

where $0 < \delta < \alpha - 1/2$. If the Riemann hypothesis holds, then $\zeta(\sigma+it) = O(t^\varepsilon)$ as well as $1/\zeta(\sigma+it) = O(t^\varepsilon)$. So the first and third integrals are $O(T^{-1+\varepsilon}x^2)$ and the second is $O(x^{1/2-\alpha+\delta}T^\varepsilon)$. Putting this all together we recover that $L_\alpha(x)$ is equal to

$$\frac{\zeta(2\alpha)}{\zeta(\alpha)} + O(x^{1/2-\alpha+\delta}T^\varepsilon) + O(T^{-1+\varepsilon}x^2)$$

Letting $T = x^3$ produces the estimate from the theorem 4.1. Conversely, if the function L_α converges for α in the desired range, then $L_s(x)$ converges uniformly in the half plane $\sigma \geq \sigma_0 > 1/2$ and so it is an analytic function in this region. The limit of which is $\zeta(2\alpha)/\zeta(\alpha)$ when $\sigma > 1$. We verify using the Euler product for the zeta function, so it must also be true for $\sigma > 1/2$. $\square$

Theorem 4.2. [5] *If the Riemann hypothesis holds, then for all* $\alpha \in [0, 1/2]$ *and all* $\epsilon > 0$*,*

$$L_\alpha(x) = O(x^{\frac{1}{2}-\alpha+\epsilon})$$

Conversely, if this estimate holds for some $\alpha \in [0, 1/2]$*, then the Riemann hypothesis is true.*

Proof. See, e.g., [5], Theorem 2.2, p.160. $\square$

Now, let's try to prove that constancy in the sign of $L_\alpha(x)$ implies both the Riemann hypothesis and the simplicity of the zeros of the zeta function. For $0 \le \alpha \le 1$, we define $\mathcal{L}_\alpha(x)$ by

$$\mathcal{L}_\alpha(x) = \begin{cases} L_\alpha(x) & \text{if } 0 \le \alpha < 1/2 \text{ or } \alpha = 1, \\ L_\alpha(x) - \frac{\log x}{2\zeta(1/2)} & \text{if } \alpha = 1/2, \\ L_\alpha(x) - \frac{\zeta(2\alpha)}{\zeta(\alpha)} & \text{if } 1/2 < \alpha < 1. \end{cases}$$

Also, define

$$A_\alpha u = \mathcal{L}_\alpha(e^u)e^{\alpha - 1/2}u$$

for $u \ge 0$, and for complex s set

$$f_\alpha(s) = \frac{\zeta(1+2s)}{(s-\alpha+1/2)\zeta(s+1/2)}. \tag{4.3}$$

Finally,

$$F_\alpha(s) = \begin{cases} f_\alpha(s) & \text{if } 0 \le \alpha < 1/2 \text{ or } \alpha = 1, \\ f_\alpha(s) - \frac{1}{2\zeta(1/2)s^2} & \text{if } \alpha = 1/2, \\ f_\alpha(s) - \frac{\zeta(2\alpha)}{\zeta(\alpha)(s-\alpha+1/2)} & \text{if } 1/2 < \alpha < 1. \end{cases} \tag{4.4}$$

Lemma 4.3. [5] *Let $\alpha \in [0,1]$. With $F_\alpha(s)$ and $A_\alpha(u)$ defined as above,*

$$F_\alpha(s) = \int_0^\infty A_\alpha(u)e^{-su}du,$$

and this integral converges for $\sigma > 1/2$. Further, under the Riemann hypothesis, it converges for $\sigma > 0$.

Proof. Using the theorem 4.1 and theorem 4.2 to prove this lemma. See, e.g., [5], Lemma 2.3, p.161-162. □

We have the following main theorem by the lemma 4.3 and Landau's theorem.

Theorem 4.4. [5] *Suppose that $\alpha \in [0,1]$ is a fixed real number. If there exists a constant C for which $A_\alpha(u) - C$ has constant sign for all sufficiently large u, then the Riemann hypothesis follows, and all the zeros of the zeta function are simple. In addition, if there exists a constant C for which $L_{1/2}(x) - C$ has constant sign for all sufficiently large x, then the Riemann hypothesis follows, and each nontrivial zeros of the zeta function has order at most two.*

Proof. Let's choose some $\alpha \in [0,1]$ and suppose some constant C exists as above. W.L.O.G., suppose that this bound is true for $u \ge 0$. We now define a function $G_\alpha(s)$ for $\sigma > 1/2$ by

$$G_\alpha(s) = \int_0^\infty (C - A_\alpha(t))\exp(-su)dt = \frac{C}{s} - F_\alpha(s)$$

This equality follows from lemma 4.3. Since the integrand has a constant sign, we can extend its domain to the left of $\sigma = 1/2$ to the first real singularity of $C/\sigma - F_\alpha(\sigma)$ via Landau's theorem. Since $\zeta(s)$ has no zeros on $\mathbb{R}^+$, it follows from equation (4.3) and (4.4) that the function $F_\alpha(\sigma)$ will have its first real singularity when $\sigma = 0$. So, the expression above defines the analytic continuation for $\sigma > 0$. Hence, $F_\alpha(s)$ is regular for these values of σ, which implies the Riemann hypothesis. Moreover, we can estimate

$$|G_\alpha(s)| \le \frac{C}{\sigma} - F_\alpha(\sigma)$$

The function $F_\alpha(s)$ has only simple poles as $\sigma \to 0^+$ from which the right-hand sign of the expression is $O(1/\sigma)$. Thus, $G(s)$ and $F(s)$ have no multiple poles when $\sigma = 0$ which means that all the zeros of the form $1/2 + i\gamma_n$ of $\zeta(s)$ are simple. To see that the order of the nontrivial zeros will be at most of order two, we argue that if $L_{1/2}(x) - C$ has a constant sign for $x \geq 1$ then when $\sigma > 1/2$, we find

$$\int_0^\infty (C - L_{1/2}(\exp t)) \exp(-st) dt = \frac{1}{2\zeta(1/2)s^2} + \frac{C}{s} - F_{1/2}(s)$$

Again, via Landau's theorem, we conclude the Riemann hypothesis, but the pole of order two at $s = 0$ in $f_{1/2}(s)$ implies that the expression above is $O(1/\sigma^2)$ as $\sigma \to 0^+$. Thus, no zeros on the critical line are of multiplicity greater than two. □

5. Special case of $L_\alpha(x)$

In the last section, we noted that there is some correspondence between the class of functions $L_\alpha(x)$ and the Riemann hypothesis. Here, we present certain conjectures involving $L_\alpha(x)$ when $\alpha \in [0, 1]$.

- In 2010, Timothy Trudgian[10] conjectured that when $\alpha = 1/2$ such that

$$L_{\frac{1}{2}}(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^{\frac{1}{2}}} \leq 0, \tag{5.1}$$

  for $17 \leq x \leq 10^{12}$ (shown in figure 5.1).

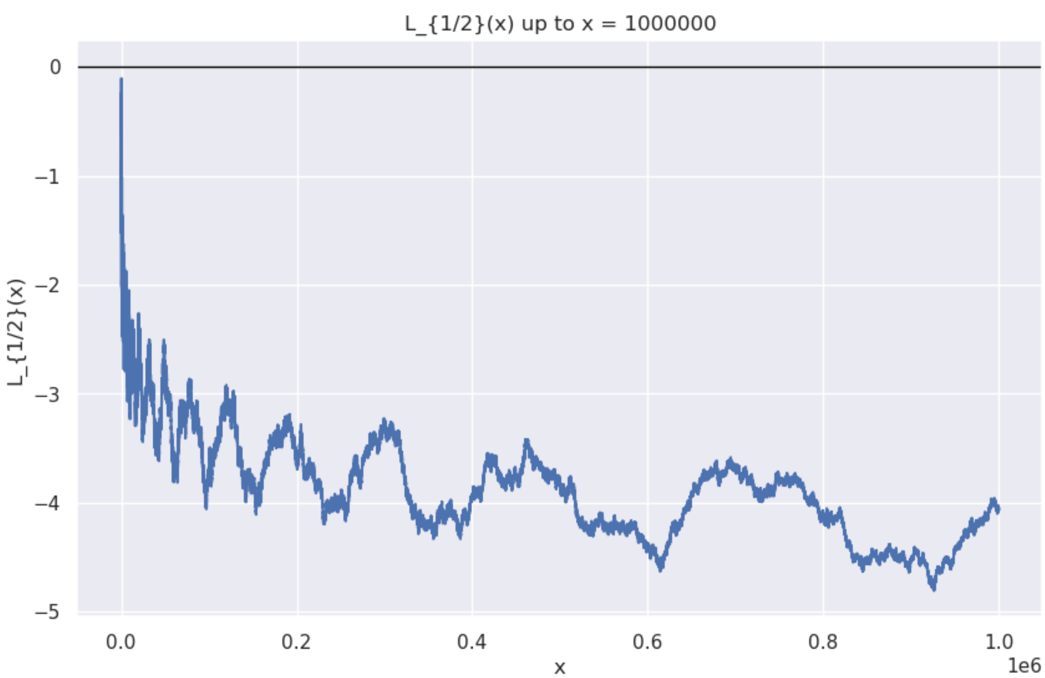

FIGURE 5.1. The functions $L_{\frac{1}{2}}(x)$ and conjectured upper bound

- In 2012, Timothy Trudgian and Michael Mossinghoff [5] conjectured that when $\alpha = 1/4$ such that

$$L_{\frac{1}{4}}(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^{\frac{1}{4}}} < 0, \tag{5.2}$$

for $11 \leq x \leq 10^{12}$ (shown in figure 5.2).

Also, they make the conjecture when $\alpha = 3/4$ such that

$$L_{\frac{3}{4}}(x) = \sum_{n \leq x} \frac{\lambda(n)}{n^{\frac{3}{4}}} > \frac{\zeta(3/2)}{\zeta(3/4)} = -0.758161736\ldots, \tag{5.3}$$

for $1 \leq x \leq 10^{12}$ (showed in figure 5.3).

Note: The smallest x where $L_{\frac{3}{4}}(x) < \zeta(3/2)/\zeta(3/4)$ is 835,018,639,060.

As we can see via the figures below, it seems plausible that these conjectures might be true. In fact, it has been shown that they are valid in the given ranges of x via computational methods. We conclude this section via a natural question that arises from this type of investigation:

Problem 5.1. Determine the smallest nontrivial value of x for each $\alpha \in [0, 1/2)$ where $L_\alpha(x) > 0$ and for $\alpha \in [1/2, 1]$ where $L_\alpha(x) < 0$.

Remark 5.2. It follows that any nontrivial sign crossing in problem 5.1 for these values of α must occur for $x > 10^{12}$.

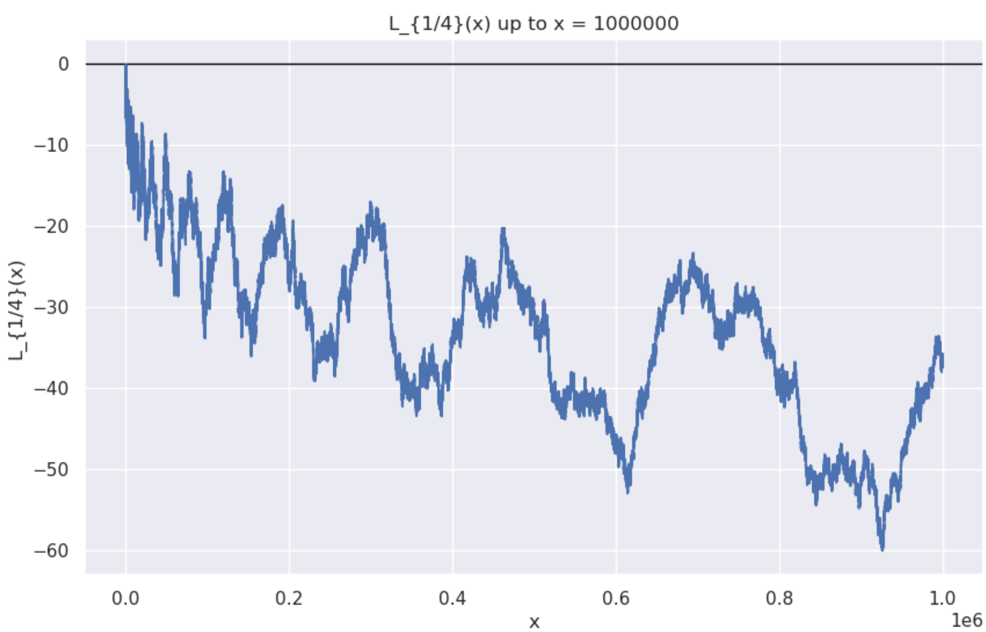

FIGURE 5.2. The functions $L_{\frac{1}{4}}(x)$ and conjectured upper bound

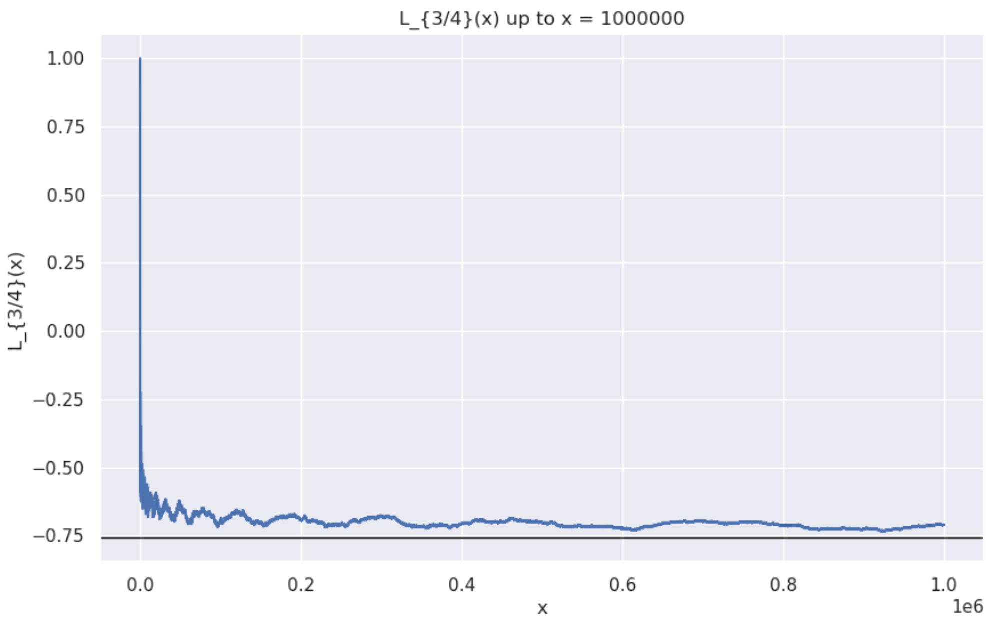

FIGURE 5.3. The functions $L_{\frac{3}{4}}(x)$and conjectured lower bound

6. CONCLUSION

The summatory Liouville function and $T(x)$, which are variants of $L_\alpha(x)$, play a role in understanding the distribution of prime numbers and the behaviour of the zeta function. Analytic properties of these functions, such as logical continuation, functional equations, and the behaviour of their zeros, can provide insights into the Riemann hypothesis. For example, $T(x)$, a smoothed version of the summatory Liouville function, has been used to investigate the correlation between the zeros of the Riemann zeta function and the distribution of primes. Additionally, the connection between $L_\alpha(x)$ functions and the Riemann hypothesis is reinforced through the study of the relationships between the Liouville function, the Möbius function, and the distribution of primes, all of which are deeply intertwined with the Riemann zeta function and the behaviour of its zeros.

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