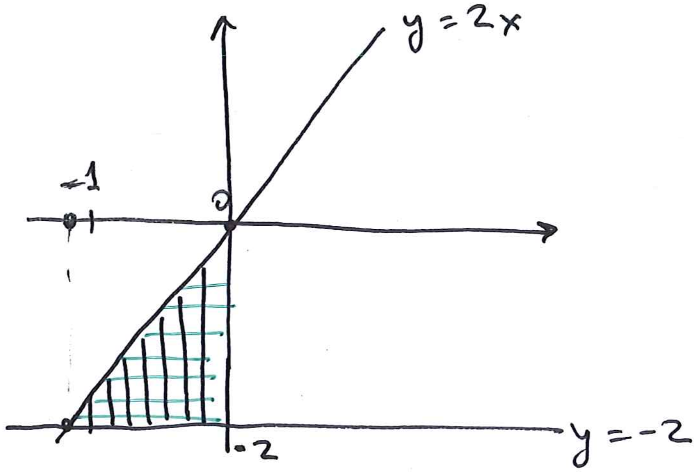
Today: Integrals over regions.

1) Interchanging the order of integration.

Example: $\int_{-1}^{0} \int_{-2}^{2x} e^{y^2}\, dy\, dx$ - evaluate this integral.

Catch: e^{y^2}: its antiderivative does not have a formula in terms of powers, exponentials, ... (elementary functions).

Try changing the order!



Answer:

$$\int_{-2}^{0} \int_{y/2}^{0} e^{y^2}\, dx\, dy$$

recall: $y = 2x$
solve for x:
$x = \frac{y}{2}$
(want x to be a function of y)

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$$= \int_{-2}^{0} \int_{y/2}^{0} e^{y^2}\,dx\,dy \;=\; \int_{-2}^{0} e^{y^2}\cdot\left(0 - \frac{y}{2}\right) dy$$

$$= -\frac{1}{2}\int_{-2}^{0} \underset{\uparrow}{\underline{\underline{y}}}\cdot e^{y^2}\,dy \;=\; -\frac{1}{2}\int_{4}^{0} \frac{1}{2}\, e^{u}\,du$$

this helps! Now we can do substitution: $u = y^2$

$du = 2y\,dy$

$$= +\frac{1}{4}\int_{0}^{4} e^{u}\,du \;=\; \frac{1}{4}\left(e^{4}-1\right)$$

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Note: variation of this example: (improper integral):

$$\int_{-1}^{0}\int_{-2}^{2x} \frac{e^y}{y}\,dy\,dx$$; looks like the same problem, $\frac{e^y}{y}$ cannot be integrated in elementary functions,

if we switch the order of integration, get extra "y", all seems fine.

But this solution would be incorrect!

But this integral is improper ($\frac{e^y}{y} \to -\infty$ at the tip  of our domain)

integral doesn't converge

you cannot change the order!

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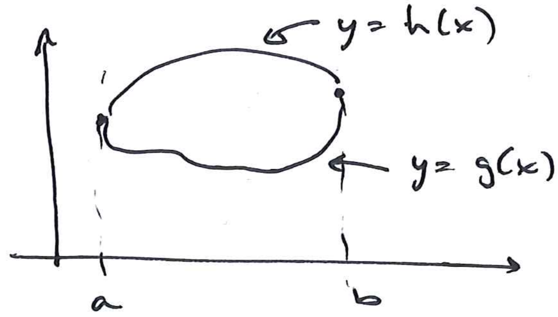
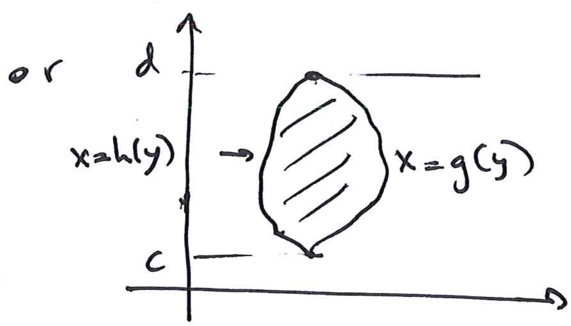
② Polar coordinates

- So far, we can deal with domains bounded by graphs of functions:



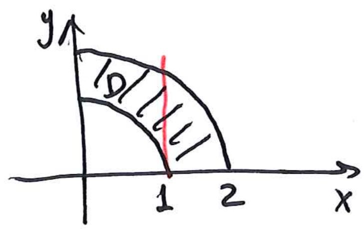
$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx$$

or



$$\int_c^d \int_{h(y)}^{g(y)} f(x,y)\, dx\, dy.$$

- what if the domain is bounded by arcs of circles? It might be inconvenient to deal with the functions:

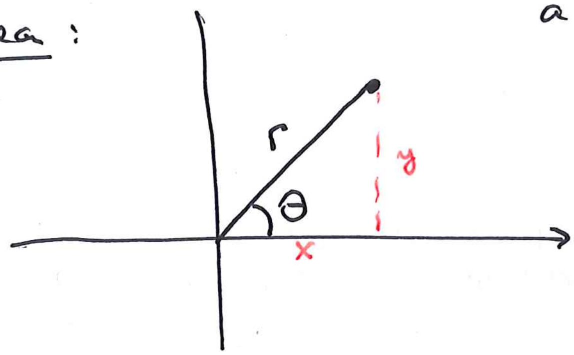


$$\iint_D f(x,y)\, dA$$

- can be done in cartesian (xy-) coordinates, but complicated.

Polar coordinates (Read 9.4) and 13.3

idea:



a point on the plane can be specified by:

- its (x,y) coords.
- distance from $(0,0)$ and direction:

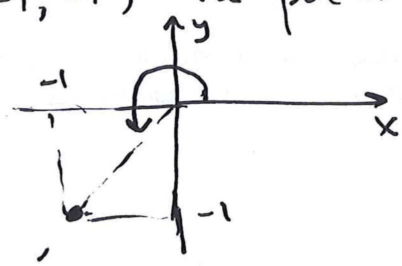
θ - angle from the positive x-axis

r = distance from $(0,0)$

$0 \leq \theta < 2\pi$

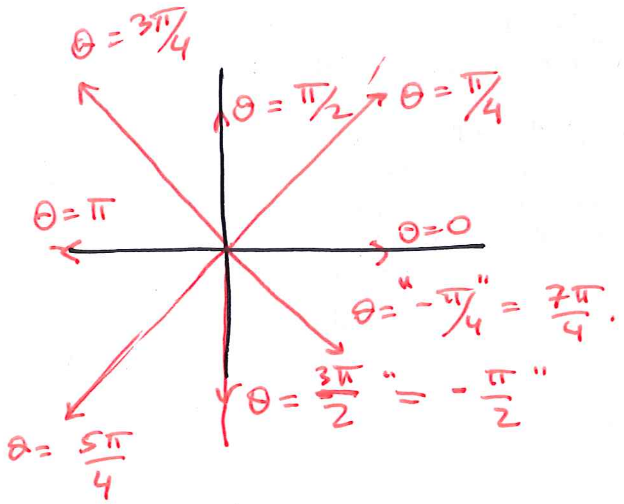
$r \geq 0$.

Example: $(-1,-1)$ in polar coordinates has expression:



$\theta = \frac{5\pi}{4}$,

$r = \sqrt{2}$



Conversion formulas

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \end{aligned}}$$

$\theta = \arctan\frac{y}{x}$ ← if you are in the upper half plane

$= \pi + \arctan\frac{y}{x}$

↑ in the lower half-plane.

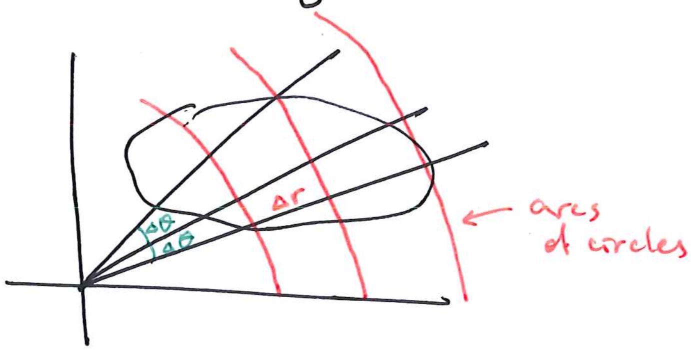
Area in polar coordinates

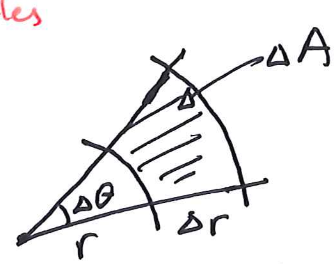
(want to change (x,y) to polar: (r, θ) in our double integrals).

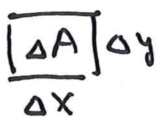
For this, need to express "dA" in polar coords:

recall: the def. of integral was based on cutting the domain into small rectangles.

Now we want to cut into "wedges":







$$\Delta A = \Delta x \Delta y$$
$$= \Delta y \Delta x$$

Need to express ΔA in terms of r, Δr, $\Delta\theta$.

Turns out: $\Delta A \approx r \cdot \Delta r \cdot \Delta\theta$