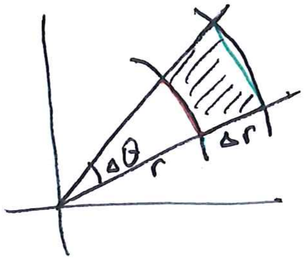
Today: • integration in polar coordinates

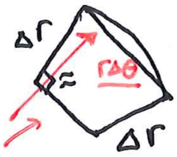
• Mass, centre of mass.

Recall last time:



$\Delta A \approx r \, \Delta r \, \Delta \theta$

Why this works: approximate arcs of circles by straight lines:



pretend the angles are straight.

pretending they are equal.

both approximately = length of the arc of a circle of angular measure $\Delta\theta$, and radius r.

$$\boxed{\Delta A \approx (\underline{r}\,\Delta\theta)\cdot\Delta r}$$

recap about π and lengths of arcs:

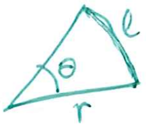
$\theta = 2\pi$ ⟷ length $= 2\pi r$

$\theta = \pi$ ⟷ half a circle, length $= \pi r$

$\theta = \frac{\pi}{2}$ ⟷ quarter of a circle, length $= \frac{\pi}{2} r$

θ ⟷ length $\ell = \theta \cdot r$

↑ in radians!

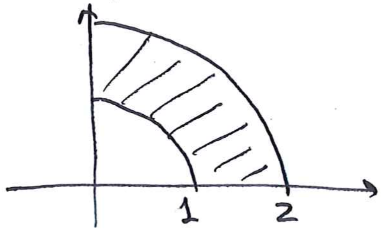


1 radian - angular measure of the angle giving arc length = 1 on the circle of radius 1).

The point: $$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta$$

$\uparrow$ plug in $x = r\cos\theta$
$y = r\sin\theta$

Example



came from $\underline{\underline{dA}}$.

Find $$\iint_D (x^2+y^2)\,dA = \int_1^2 \int_0^{\pi/2} r^2 \cdot \underline{\underline{r}}\,d\theta\,dr$$

$$= \frac{\pi}{2} \cdot \int_1^2 r^3\,dr = \frac{\pi}{2} \cdot \frac{r^4}{4}\Big|_1^2$$

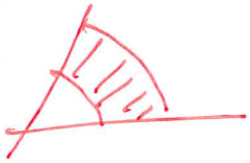
$$= \frac{\pi}{2} \cdot \frac{1}{4} \cdot 15$$

D: $1 \le r \le 2$
$0 \le \theta \le \pi/2$

recall: $r = \sqrt{x^2+y^2}$
$x^2+y^2 = r^2 \; (= r^2\cos^2\theta + r^2\sin^2\theta)$

Note: earlier, when writing iterated integrals in xy-coordi,

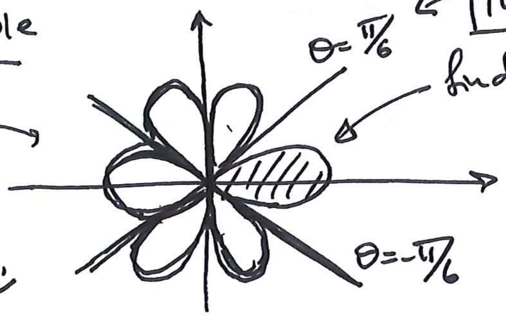
- we discussed: constant limits ↔ domain is a rectangle
- in polar coordinates, both limits constant ↔
  - between two circles
  - between two rays.



Application : mass and centre of mass. (next time).

First: total area: how do you find area of a region?

Example

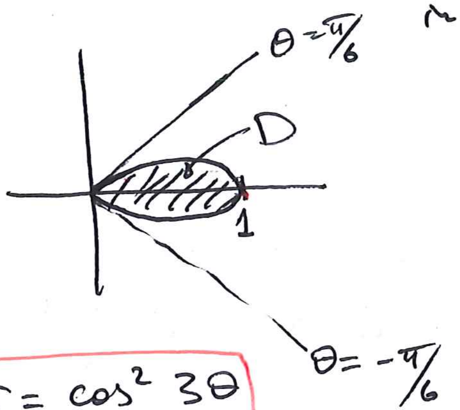


these lines are tangent to the petals

find the area.

all the petals should be equal ☺

(see 'gallery of polar curves' p. 533 in the book)



$\boxed{r = \cos^2 3\theta}$

- equation of the flower.

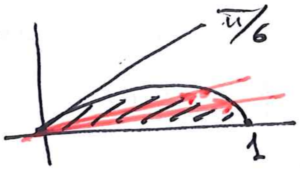
(see 9.4).

① $\boxed{\text{Area} = \iint_D 1\, dA.}$

② Set up the integral in polar coordinates.

how to set up limits of integration?

1.



$0 \le \theta \le \pi/6$

↑

compute by:

$r = 0$

$\cos^2 3\theta = 0$

$3\theta = \frac{\pi}{2} + \pi k$

$\theta = \underline{\underline{\frac{\pi}{6}}} + \frac{\pi}{3} k$

2. Given:

$r = \cos^2 3\theta$

is the boundary.

So:

$0 \le r \le \cos^2 3\theta$

$$A = 2 \int_0^{\pi/6} \int_0^{\cos^2 3\theta} 1 \cdot r \, dr \, d\theta$$

$$\left(\text{also: } A = \int_{-\pi/6}^{\pi/6} \int_0^{\cos^2 3\theta} 1 \cdot r \, dr \, d\theta \right)$$

$$= 2 \int_0^{\pi/6} \frac{r^2}{2} \Big|_0^{\cos^2 3\theta} d\theta = \int_0^{\pi/6} \cos^4(3\theta) \, d\theta$$

$$\Bigg[\underline{\text{Recall}}: \quad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\cos^4(3\theta) = \left(\cos^2(3\theta)\right)^2 = \left(\frac{1+\cos(6\theta)}{2}\right)^2$$

$$= \frac{1}{4} + \frac{1}{2}\cos(6\theta) + \cos(6\theta)^2 \cdot \frac{1}{4}$$

$$= \frac{1}{4} + \frac{1}{2}\cos(6\theta) + \frac{1}{4} \cdot \frac{1}{2}\left(1 + \cos(12\theta)\right) \Bigg]$$

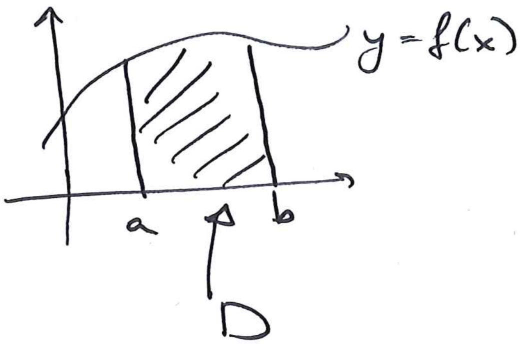
Then $$\int_0^{\pi/6} \cos^4(3\theta) \, d\theta = \frac{1}{4} \int_0^{\pi/6} (1 + 2\cos(6\theta) + \frac{1}{2}\cos(12\theta) d\theta$$

$$= \frac{1}{4} \cdot \left(\frac{\pi}{6} + \underset{\overset{\shortparallel}{0}}{2\sin(6\theta) \cdot \frac{1}{6} \Big|_0^{\pi/6}} + \frac{1}{2} \cdot \underset{\overset{\shortparallel}{0}}{\sin(12\theta) \cdot \frac{1}{12} \Big|_0^{\pi/6}} \right)$$

$$= \boxed{\frac{\pi}{24}}.$$

Note: how this way of finding area relates to calculus-101:

there we had:



area under the graph of $y=f(x)$

$$= \int_a^b f(x)\,dx .$$

What if we use today's method to compute this area? Let D be the shaded domain under the graph (see picture).

Then today's formula says:

$$\text{area of } D = \iint_D 1\, dA \underset{\uparrow}{=} \int_a^b \int_0^{f(x)} 1\, dy\, dx$$

encode the domain D into the limits of integration

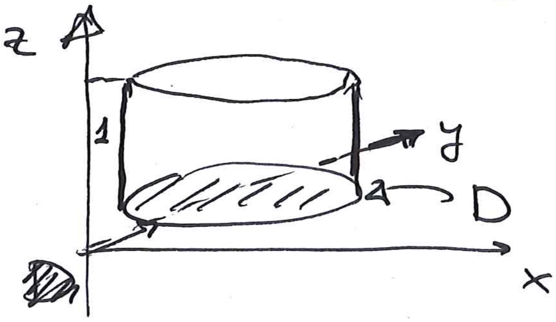
$$\underset{\uparrow}{=} \int_a^b f(x)\,dx$$ — of course, agrees with Math 101.

evaluate the inner integral

The point of today's formula is that now we can find areas of more general domains than the domain under the graph of a positive function. (such as the flower from the example)

Note also: generally, $\iint_D f(x,y)\, dA$ is the volume under the graph of $f(x,y)$, if $f(x,y)$ is a positive function.

What if we take $f(x,y) \equiv 1$? Then its graph is the horizontal plane, and the volume under the graph is the volume of the cylinder with base D and height 1:



This volume = (height)(area of base)

= (area of D) $\cdot$ 1,

so we get (again)

$$\text{volume} = \iint_D 1\, dA = \text{area of } D.$$

Finally, some people asked, why can't we use the formula for the area under the graph of the function $\cos^2(3\theta)$ to find the area of the flower in the example. Note the difference: $r = \cos^2(3\theta)$ is an equation in polar coordinates, it does NOT define a graph of $y = \cos^2(3x)$!

If you want to think of it as a graph, convert to xy-coordinates: $\sqrt{x^2+y^2} = \cos^2(3 \arctan \frac{y}{x})$

– this gives y as an implicit function of x, but