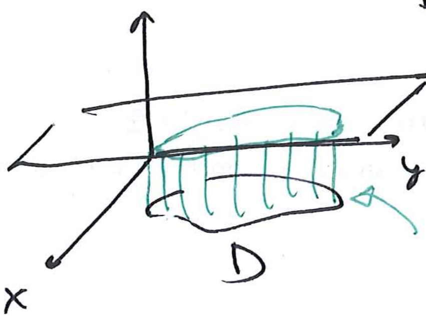
Recall: Area of $D = \iint_D 1 \, dA$

Why:



$z=1$ – graph of the function $f(x,y)=1$.

volume over D under the graph $= \iint_D 1 \, dA$

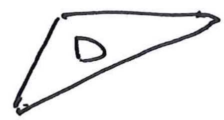
$= \begin{pmatrix} \text{base} \\ \text{area} \end{pmatrix} \cdot \text{height}$, where $\begin{pmatrix} \text{base} \\ \text{area} \end{pmatrix} = \begin{pmatrix} \text{area} \\ \text{of } D \end{pmatrix}$ and $\text{height} = 1$

Mass and centre of mass

1) Total mass.

imagine a metal plate of variable density:

mathematical model for it:



a domain D with

<u>density function</u> $\rho(x,y)$.

What is <u>density</u>?

– mass/volume – 3^d-density.

<u>Here</u>: $\dfrac{\text{mass}}{\text{area}}$ ← thin plate density (2^d)

$\dfrac{\text{mass}}{\text{length}}$ ← for a <u>wire</u> (1^d).

Refinement: when density is not constant,

$$\frac{\text{total mass}}{\text{total area}} = \text{average density};$$

but to compute density around a point (x_0, y_0), we take small square around (x_0, y_0), compute

$$\rho(x_0, y_0) = \lim_{\text{size of } \square \to 0} \frac{\text{mass}(\square)}{\text{area}(\square)}$$

So: $$\boxed{\text{total mass of the plate} = \iint_D \rho(x,y)\, dA}$$

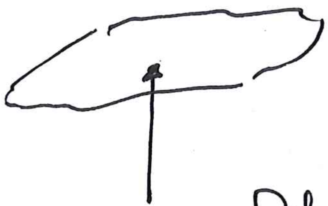
units of mass / units of area

Note: if density $\rho(x,y)$ is constant $= 1 \frac{g}{mm^2}$

$$\text{total mass of a lamina} = \iint_D 1 \cdot dA = (\text{area of } D) \cdot \frac{g}{mm^2}$$

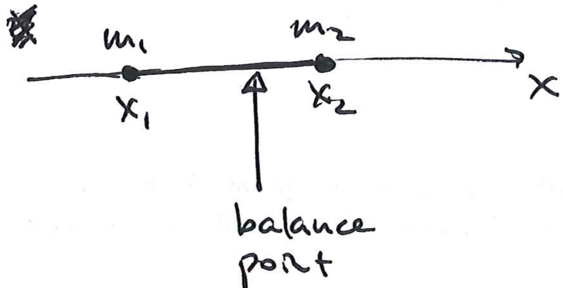
in mm^2

Centre of mass



if place a needle at the centre of mass the plate would be balanced.

Def (for a finite mass system on a line):



$$\bar{X} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

centre of mass

weighted average of x_1, x_2

Note: $= \frac{x_1 + x_2}{2}$ = midpoint if $m_1 = m_2$.

(Note: expected value in probability is the same thing).

- For $\underline{\underline{n}}$ points : $x_1, \dots x_n$ - masses $m_1, \dots m_n$

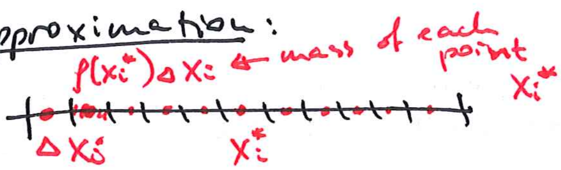
$$\bar{X} = \frac{m_1 x_1 + \dots + m_n x_n}{m_1 + \dots + m_n}$$ ← total mass.

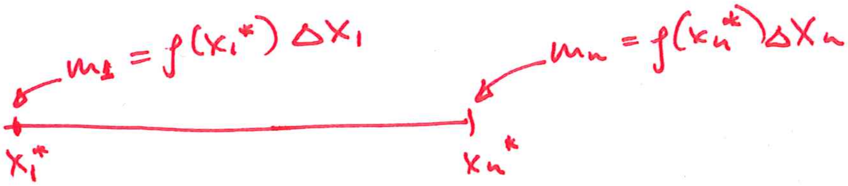
- Continuous wire of density $\rho(x)$:

centre of mass:

$$\bar{X} = \frac{\int_a^b x\, \rho(x)\, dx}{\text{total mass}}$$

approximation:





$$\left[\ \text{total mass} = \int_a^b \rho(x)\, dx.\ \right]$$

- $\underline{\text{In } 2^d}$: $\rho(x,y)$ - $\underline{\text{density}}$:

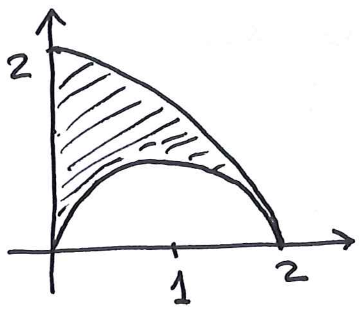
$\underline{\text{Mass}}$ $\quad M = \iint_D \rho(x,y)\, dA$

$$\bar{X} = \frac{1}{M}\underbrace{\iint_D x\,\rho(x,y)\, dA}$$ ← M_y - moment about y-axis

$$\bar{y} = \frac{1}{M}\underbrace{\iint_D y\,\rho(x,y)\, dA}$$ ← M_x = moment about x-axis.

$$\boxed{\bar{X} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}}$$

Worksheet 10



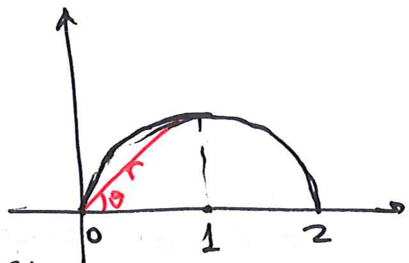
A lamina occupies the area pictured (between the circle of radius 2 centered at $(0,0)$ and the circle of radius 1 centred at $(1,0)$).

Its density is $\rho(x,y) = \sqrt{x^2+y^2}$.

Find the total mass and the y-coordinate of the centre of mass.

The difficulty:

equation of this circle in polar coordinates



to get it, start with xy:

$$(x-1)^2 + y^2 = 1$$

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$$

$$r^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1}$$

$$r^2 - 2r\cos\theta = 0$$

$$\boxed{r = 2\cos\theta}$$

$$M = \iint_D \rho(x,y)\, dA \underset{\text{in polar}}{=} \int_0^{\pi/2}\int_{2\cos\theta}^{2} \overset{\text{from } \rho}{r} \cdot \overset{\text{from } dA}{r}\, dr\, d\theta$$

$$\rho(x,y) = \sqrt{x^2+y^2} = r$$

$$= \int_0^{\pi/2}\int_{2\cos\theta}^{2} r^2\, dr\, d\theta = \frac{1}{3}\int_0^{\pi/2} r^3 \Big|_{2\cos\theta}^{2} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \left(8 - 8\cos^3\theta\right) d\theta$$

$$= \frac{8}{3}\cdot\frac{\pi}{2} - \frac{8}{3}\int_0^{\pi/2} \overbrace{(1-\sin^2\theta)\cos\theta}^{\cos^3\theta}\, d\theta \overset{u=\sin\theta}{=} \frac{4\pi}{3} - \frac{8}{3}\int_0^1 (1-u^2)\, du = \frac{4\pi}{3} - \frac{8}{3}\left(1-\frac{1}{3}\right)$$

$$= \boxed{\frac{4\pi}{3} - \frac{16}{9}}$$

this is M.

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{M}\int_0^{\pi/2}\int_{2\cos\theta}^{2} \underset{x}{(r\cos\theta)} \cdot \underset{\rho}{r} \cdot \underbrace{r\, dr\, d\theta}_{dA} \\ \bar{y} &= \frac{1}{M}\int_0^{\pi/2}\int_{2\cos\theta}^{2} (r\sin\theta)\cdot r \cdot r\, dr\, d\theta \end{aligned}}$$

Computing $\bar{x}$ and $\bar{y}$: (actually, the question was only about $\bar{y}$, but I'll do both).

$$\bar{x} = \frac{1}{M}\int_0^{\pi/2}\int_{2\cos\theta}^{2} r^3\cos\theta\, dr\, d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2} \frac{r^4}{4}\Big|_{2\cos\theta}^{2} \cdot \cos\theta\, d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2} \left(4 - 4\cos^4\theta\right)\cdot\cos\theta\, d\theta$$

$$= \frac{4}{M}\int_0^{\pi/2} \cos\theta - \cos^5\theta\, d\theta = \frac{4}{M}\cdot\sin\theta\Big|_0^{\pi/2} - \frac{4}{M}\int_0^{\pi/2}\cos^5\theta\, d\theta$$

As before, an odd power of $\cos\theta$ is integrated using the change of variable $u = \sin\theta$:

$$\int_0^{\pi/2} \cos^5\theta \, d\theta \underset{\substack{u=\sin\theta \\ du=\cos\theta\, d\theta \\ \cos^4\theta = (1-\sin^2\theta)^2}}{=} \int_0^{1} (1-u^2)^2 \, du = \int_0^1 (1-2u^2+u^4)\, du$$

$$= 1 - 2\cdot\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

Putting it together, get:

$$\bar{x} = \frac{4}{M} - \frac{4}{M}\cdot\frac{8}{15} = \boxed{\frac{4}{M}\cdot\frac{7}{15}}$$

(M as above)

$$M = \frac{4\pi}{3} - \frac{16}{9}$$

$$\bar{y} = \frac{1}{M}\int_0^{\pi/2}\int_{2\cos\theta}^{2} \underbrace{r\sin\theta}_{y} \cdot \underbrace{r}_{\rho(x,y)} \cdot r \, dr\, d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2}\left(\int_{2\cos\theta}^{2} r^3\, dr\right)\sin\theta\, d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2} \left.\frac{r^4}{4}\right|_{2\cos\theta}^{2} \sin\theta\, d\theta$$

$$= \frac{4}{M}\int_0^{\pi/2} (1-\cos^4\theta)\sin\theta\, d\theta \underset{\substack{\boxed{u=\cos\theta} \\ du=-\sin\theta\, d\theta}}{=}$$

$$= \frac{4}{M}\left(-\int_1^0 (1-u^4)\, du\right) = \frac{4}{M}\int_0^1 (1-u^4)\, du$$

$$= \frac{4}{M}\cdot\left(1-\frac{1}{5}\right) = \boxed{\frac{16}{5M}}$$

$\boxed{M \text{ as above}}$