Tangent planes to surfaces in space

For a graph of a function of two variables, we have two main approaches to writing the equation of the tangent plane at a given point. It is important to understand that these two ways agree. (There is also a third way, summarized at the end, which also gives the same answer, of course).

Here is a table summarizing the situation:

equation of the surface	point on it	A normal vector to the tangent plane
graph of $f(x,y)$: $z = f(x,y)$	$(a, b, f(a,b))$	$\langle -f_x(a,b), -f_y(a,b), 1\rangle$
equation $F(x,y,z) = 0$	(a,b,c) with $F(a,b,c) = 0$	$\nabla F\|_{(a,b,c)}$ $= \langle F_x(a,b,c), F_y(a,b,c), F_z(a,b,c)\rangle$

The point is that the first line is a special case of the second line. If our surface is the graph of $f(x,y)$, then it has the equation $z = f(x,y)$. Let us make a new function: $F(x,y,z) = z - f(x,y)$. **Note: $F(x,y,z)$ is a function of 3 variables, while $f(x,y)$ is a function of 2 variables!**

Then our graph is also given by the equation $F(x,y,z) = 0$. The gradient of F is exactly $\nabla F = \langle -f_x, -f_y, 1\rangle$.

This leads to the same equation of the tangent plane that we get from linearization: we learned earlier in this course that the tangent plane to the graph of $f(x,y)$ is given by:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

A normal vector to this plane is: $\langle -f_x(a,b), -f_y(a,b), 1\rangle$, which (of course!) agrees with the gradient ∇F.

The same equation using the cross product. We also discussed that if you take the graph $z = f(x,y)$ and consider its cross-sections with the planes $x = a$ and $y = b$, you get two curves, called the *traces* of $f(x,y)$ on these planes. The tangent vectors to these curves are: $\mathbf{v}_1 = \langle 1, 0, f_x(a,b)\rangle$ and $\mathbf{v}_2 = \langle 0, 1, f_y(a,b)\rangle$. Why so: consider for example the plane $x = a$. The trace of $f(x,y)$ on this plane is the graph of the function of y, let us call it $h(y) := f(a,y)$ that we obtain by plugging in $x = a$ into f. The slope of the tangent line to this graph at $y = b$ is $h'(b) = f_y(a,b)$. The slope is the ratio of the z-component to the y-component here; we can choose the y-component to be 1. Then we get that our tangent vector should have the z-component equal to $f_y(a,b)$. Its x-component is 0 because it lies in the plane $x = a$ (a vertical plane parallel to the y-axis; all vectors in it have zero x-component). Thus we get the vector $\langle 0, 1, f_y(a,b)\rangle$.

Now the tangent plane to the graph $z = f(x,y)$ at (a,b) must contain both these tangent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, so we can use the usual method for finding an equation of a plane containing a given point and parallel to two given vectors. Its normal is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. Computing the cross product, we get:

$$\mathbf{n} = \langle 1, 0, f_x(a,b) \rangle \times \langle 0, 1, f_y(a,b) \rangle = \langle -f_x(a,b), -f_y(a,b), 1 \rangle.$$

This gives us the same equation of the plane as line 1 in the table above.