This test has 4 questions on 6 pages, for a total of 30 points.

Duration: 45 minutes

- Write your name on every page.
- **you need to show enough work to justify your answers.**
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. A one-sided cheat sheet is allowed. **Electronic devices of any kind (including calculators, cell phones, etc.) are NOT allowed.**

Full Name (including all middle names): Solutions.

Student-No: ______

Signature: ______

Section number/time of class: ______

Name of the instructor: ______

Question:	1	2	3	4	Total
Points:	9	5	6	10	30
Score:					

1. In a lab experiment, a small fish is placed at the point with coordinates $(1, 2, 3)$ in a tank with salt water. The salinity of the water at the point with coordinates (x, y, z) is given by the formula

$$f(x, y, z) = 35(1 - e^{-2x-y-z}) \text{ parts per thousand .}$$

2 marks (a) Find the gradient of f at the point $(1, 2, 3)$.

$f_x = 35 \cdot 2e^{-2x-y-z}$
$f_y = 35e^{-2x-y-z}$
$f_z = 35e^{-2x-y-z}$

at $(1,2,3)$:

$\boxed{\bar{\nabla} f|_{(1,2,3)} = \langle 2, 1, 1\rangle \cdot 35e^{-7}}$

2 marks (b) Find the directional derivative of $f(x, y, z)$ in the direction of the vector $\langle 3, 4, 5\rangle$.

$\bar{u} = \frac{1}{\sqrt{9+16+25}} \langle 3,4,5\rangle = \left\langle \frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}} \right\rangle$

– unit vector in the direction of $\langle 3,4,5\rangle$

$D_{\bar{u}} f = \bar{\nabla} f \cdot \bar{u} = 35e^{-7} \cdot \frac{1}{5\sqrt{2}} (2 \cdot 3 + 4 + 5)$

$= \boxed{\frac{7e^{-7}}{\sqrt{2}} \cdot 15}$

2 marks (c) In which direction should the fish swim from the point $(1, 2, 3)$ to experience the fastest *decrease* of salinity?

opposite to the gradient, that is, in the direction of the vector $\langle -2, -1, -1\rangle$

[any vector in the correct direction is accepted as the right answer here]

3 marks (d) Find the rate of change of salinity (in parts per thousand/sec) the fish will experience as it starts swimming from the point $(1, 2, 3)$ with velocity $\bar{v} = \langle 3, 4, 5\rangle$ (where the components are measured in cm/sec).

By chain rule, $\frac{df}{dt} = \bar{\nabla} f \cdot \bar{v} = 35e^{-7} \cdot 15 =$

$= \boxed{525e^{-7}}$

This page has been left blank for your workings and solutions.

3 marks 2. (a) Find a vector normal to the curve $x^3y - y^4x^2 = -14$ at the point $(1, 2)$.

2 marks (b) Find the equation of the tangent line to this curve at the point $(1, 2)$.

Let $f(x,y) = x^3y - y^4x^2 + 14$, $f_x = 3x^2y - 2y^4x$

$\nabla f\big|_{(1,2)} = \langle -26, -31 \rangle$ $f_y = x^3 - 4y^3x^2$

and this vector is normal to the curve.

a): $\boxed{\langle -26, -31 \rangle}$

b): $\boxed{-26(x-1) - 31(y-2) = 0.}$

6 marks 3. Find and classify the critical points of

$$f(x,y) = xy^2 - y - 4x + 6.$$

$f_x = y^2 - 4$

$f_y = 2xy - 1$

$\begin{cases} y^2 - 4 = 0 \\ 2xy - 1 = 0 \end{cases}$

$y = \pm 2$

$y = 2$: $4x - 1 = 0$

$x = 1/4$

$y = -2$: $-4x - 1 = 0$

$x = -1/4$

Critical points: $\boxed{\left(\frac{1}{4}, 2\right) \quad \left(-\frac{1}{4}, -2\right)}$

$f_{xx} = 0$

$f_{xy} = 2y$

$f_{yy} = 2x.$

$D = \begin{vmatrix} 0 & 2y \\ 2y & 2x \end{vmatrix} = -4y^2 < 0$

always. (if $y \neq 0$)

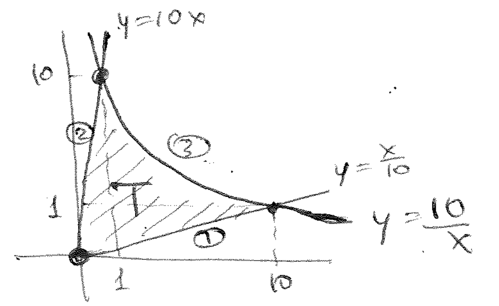
The both critical points are $\boxed{\text{saddles}}$

4. Let $f(x,y) = xy^2 - y - 4x + 6$ (the same function as in the previous problem). Let T be the (closed) region bounded by the lines $y = 10x$, $y = \frac{x}{10}$, and the hyperbola $y = \frac{10}{x}$.

2 marks (a) Sketch T.

8 marks (b) List all the points in T where absolute maximum or minimum of f could occur (do **not evaluate the function at these points**).

a)



Answer: $\left(\frac{1}{4}, 2\right)$, $\left(\frac{\sqrt{14}}{10\sqrt{3}}, \frac{\sqrt{14}}{\sqrt{3}}\right)$,

$(0,0)$, $(10,1)$, $(1,10)$ from edge ②

vertices

(b) 1. Critical points in T: from problem ③, the critical points are $\left(-\frac{1}{4}, -2\right)$ and $\left(\frac{1}{4}, 2\right)$

The point $\boxed{\left(\frac{1}{4}, 2\right)}$ lies in T

2. boundary: ① the line $y = \frac{x}{10}$, $0 \le x \le 10$

$$g_1(x) = f\left(x, \frac{x}{10}\right) = x \cdot \frac{x^2}{100} - \frac{x}{10} - 4x + 6$$

$$= \frac{x^3}{100} - \frac{41}{10}x + 6.$$

$$g_1'(x) = \frac{1}{100} \cdot 3x^2 - \frac{41}{10} \qquad g_1'(x) = 0: \quad \frac{3x^2}{10} - 41 = 0$$

$$x = \pm\sqrt{\frac{410}{3}}$$

– outside $[0,10]$

② The line $y = 10x$, $0 \le x \le 1$

$$g_2(x) = f(10x, x) = 100x^3 - 10x - 4x + 6$$

$$g_2'(x) = 300x^2 - 14 \qquad x = \pm\sqrt{\frac{14}{300}} = \frac{1}{10}\sqrt{\frac{14}{3}}$$

get a critical point $\boxed{\left(\frac{1}{10}\sqrt{\frac{14}{3}}, \sqrt{\frac{14}{3}}\right)}$

③ The hyperbola $y = \frac{10}{x}$, $1 \le x \le 10$

$$g_3(x) = f\left(x, \frac{10}{x}\right) = x \cdot \frac{100}{x^2} - \frac{10}{x} - 4x + 6 = \frac{90}{x} - 4x + 6$$

$$g_3'(x) = -\frac{90}{x^2} - 4 \neq 0.$$