1. In a lab experiment, an amoeba is placed at the point with coordinates $(1,2)$ in a shallow square dish with salt water. The salinity of the water at the point with coordinates (x,y) is given by the formula

$$f(x,y) = 35(1 - e^{-x-2y}) \text{ parts per thousand .}$$

2 marks

(a) Find the gradient of f at the point $(1,2)$.

$f_x = 35 \cdot e^{-x-2y}$

$f_y = 35 \cdot 2e^{-x-2y}$

at $(1,2)$: $\boxed{\bar{\nabla} f\big|_{(1,2)} = \langle 1, 2 \rangle \cdot 35e^{-5}}$

2 marks

(b) Find the directional derivative of $f(x,y)$ in the direction of the vector $\langle 3,4 \rangle$.

$\bar{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$ – unit vector in the direction $\langle 3,4 \rangle$

$D_{\bar{u}} f = \bar{\nabla} f \cdot \bar{u} = 35e^{-5} \cdot \langle 1,2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$

$= \boxed{77e^{-5}}$

2 marks

(c) In which direction should the amoeba swim from the point $(1,2)$ to experience the fastest *decrease* of salinity?

Fastest decrease of $f(x,y)$ is the direction opposite to the gradient.

Answer: $\boxed{\langle -1, -2 \rangle}$ (any vector in this direction is accepted as the right answer)

3 marks

(d) Find the rate of change of salinity (in parts per thousand/sec) the amoeba will experience as it starts swimming from the point $(1,2)$ with velocity $\bar{v} = \langle 3,4 \rangle$ (where the components are measured in cm/sec).

$\frac{df}{dt} = \bar{\nabla} f \cdot \bar{v} = 35e^{-5} \langle 1,2 \rangle \cdot \langle 3,4 \rangle$

$= \boxed{385e^{-5}}$ cm/sec

5 marks 2. Find the equation of the tangent plane to the surface $x^3z - y^4x^2 + z^5y = -16$ at the point $(1, 2, 0)$.

let $f(x,y,z) = x^3z - y^4x^2 + z^5y + 16$

The gradient at this point is normal to the tangent plane.

We get: $\vec{\nabla} f\Big|_{(1,2,0)} = \langle -32, -32, 1 \rangle$

$f_x = 3x^2z - 2y^4x$

$f_y = -4y^3x^2 + z^5$

$f_z = x^3 + 5z^4y$

Answer: $\boxed{-32(x-1) - 32(y-2) + 1\cdot(z-0) = 0}$

6 marks 3. Find and classify the critical points of

$$f(x,y) = x^2y - x - 4y + 6.$$

$f_x = 2xy - 1$
$f_y = x^2 - 4$

Get: $\begin{cases} 2xy - 1 = 0 \\ x^2 - 4 = 0 \end{cases}$ $x = \pm 2$

if $x = 2$: $4y - 1 = 0$

if $x = -2$: $-4y - 1 = 0$

Get the points: $\boxed{\left(2, \frac{1}{4}\right) \text{ and } \left(-2, -1/4\right)}$

classification:

$f_{xx} = 2y$
$f_{xy} = 2x$
$f_{yy} = 0$

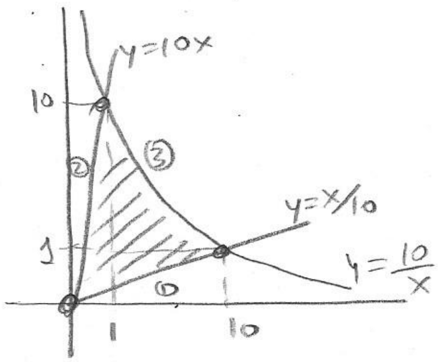
$$D = \begin{vmatrix} 2y & 2x \\ 2x & 0 \end{vmatrix} = -4x^2 < 0$$
if $x \neq 0$.

Answer: both points are saddle points

4. Let $f(x, y) = x^2y - x - 4y + 6$ (the same function as in the previous problem). Let T be the (closed) region bounded by the lines $y = 10x$, $y = \frac{x}{10}$, and the hyperbola $y = \frac{10}{x}$.

2 marks (a) Sketch T.

8 marks (b) List all the points in T where absolute maximum or minimum of f could occur (**do not evaluate the function at these points**).



Answer: $(2, 1/4)$
$\left(\frac{\sqrt{14}}{\sqrt{3}}, \frac{\sqrt{14}}{10\sqrt{3}}\right)$,
$(0,0)$, $(1,10)$, $(10,1)$.

(1) From the previous problem, we have one critical point $(2, \frac{1}{4})$ inside the domain
[though since we know it is a saddle, it is not relevant]

(2) Boundary: piece (1): $y = \frac{x}{10}$, $0 \leq x \leq 10$

Let $g_1(x) = f(x, \frac{x}{10}) = x^2 \cdot \frac{x}{10} - x - 4 \cdot \frac{x}{10} + 6$

$= \frac{x^3}{10} - \frac{14x}{10} + 6$

$g_1'(x) = \frac{1}{10}(3x^2 - 14)$

$g_1'(x) = 0$: $x = \pm\frac{\sqrt{14}}{\sqrt{3}}$, get: $\left(\frac{\sqrt{14}}{\sqrt{3}}, \frac{\sqrt{14}}{10\sqrt{3}}\right)$ ← possible point.

piece (2) $y = 10x$, $0 \leq x \leq 1$

$g_2(x) = f(x, 10x) = x^2 \cdot 10x - x - 4 \cdot 10x + 6 = 10x^3 - 41x + 6$

$g_2'(x) = 30x^2 - 41x$ $g_2'(x) = 0$: $x = \pm\sqrt{\frac{41}{30}}$ — outside of $[0,1]$.

piece (3) $g_3(x) = f(x, \frac{10}{x}) = x^2 \cdot \frac{10}{x} - x - 4 \cdot \frac{10}{x} + 6 = 9x - \frac{40}{x}$

$g_3'(x) = 9 + \frac{40}{x^2} > 0$ on $[1, 10]$