Some harder problems on cardinality.

These are two series of problems with specific goals: the first goal is to prove that the cardinality of the set of irrational numbers is *continuum*, and the second is to prove that the cardinality of $\mathbb{R} \times \mathbb{R}$ is continuum, without using Cantor-Bernstein-Schröeder Theorem.

1. Does there exist a continuous bijective function $f : \mathbb{R} \to \mathbb{R} - \{1\}$? Explain.
   Hint: Recall the Intermediate Value Theorem.

> **Solution:** The answer is "No". Suppose $f : \mathbb{R} \to \mathbb{R} - \{1\}$ is bijective. Then there exists $a \in \mathbb{R}$, such that $f(a) = 0$, and there exists $b \in \mathbb{R}$, such that $f(b) = 2$. Then, since f is continuous, by the Intermediate Value Theorem, there exists $c \in (a, b)$, such that $f(c) = 1$. Thus, the range of such a continuous function has to contain 1.

2. Prove that $|(0, 1)| = |[0, 1)|$.
   Hint. Do not try to write a formula for a bijective function. Instead, choose an infinite countable subset A of $(0, 1)$. Then make a bijection between A and $A \cup \{0\}$. Then define your function on the rest of the interval. The resulting function will be very discontinuous, but that does not matter!

> **Solution:** Lemma. If $f : A \to B$ and $g : C \to D$ are bijections where $A \cap C = B \cap D = \emptyset$, then $h : A \cup C \to B \cup D$ defined by
>
> $$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C. \end{cases}$$
>
> is a bijection.
>
> Proof of Lemma. Assume $h(x) = h(x') = y$. Assume $y \in B$. Then x, x' must be in A since h maps C into the disjoint set D. Therefore $f(x) = f(x')$ and so $x = x'$ as f is injective. A similar argument works if $y \in D$. Hence h is injective. Now let $y \in B$. Then there is an $x \in A$ so that $h(x) = f(x) = y$. Therefore $B \subset \text{ran}(h)$. Similarly $D \subset \text{ran}(h)$. We are done.
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> Proof of Question 5. Let $a_n = (n + 1)^{-1} \in (0, 1)$ for $n \in \mathbb{N}$ and $A = \{a_n : n \in \mathbb{N}\}$. Define $f : A \to A \cup \{0\}$ by
>
> $$f(a_1) = 0 \text{ and } f(a_n) = a_{n-1} \text{ if } n \geq 2.$$
>
> Assume $f(a_i) = f(a_j) = y$. If $y = 0$, then $a_i = a_j = a_1$. If $y = a_k \in A$, then $a_i = a_j = a_{k+1}$. Hence f is injective. For any a_n, $a_n = f(a_{n+1}) \in \text{ran}(f)$ and $0 = f(a_1)$, so f is onto. Hence f is bijective. Now define let g be the identity function on $(0, 1) - A$ (clearly a bijection). Defining $h : (0, 1) \to [0, 1)$ as in the Lemma above we see that h is a bijection (by the Lemma) and so $|(0, 1)| = |[0, 1)|$.

3. Let A be any uncountable set, and let $B \subset A$ be a countable subset of A. Prove that $|A| = |A - B|$.
Hint. This is a generalization of the previous problem, and has a very similar solution.

Solution: Let $C = \{c_1, \dots c_n, \dots\}$ be any countable infinite subset of $A - B$ (it seems obvious that we can just pick an element c_1 from the infinite set $A - B$, then pick any element c_2 from the set of remaining elements, and keep doing it; note, however, that technically, this is the *axiom of choice*, which we do not discuss in this course). Once we have the set C, we know that C and $C \cup B$ are both countably infinite, and therefore there exists a bijective function $f : C \cup B \to C$. Now, define $h : A \to A - B$ by:

$$h(x) = \begin{cases} x, & x \in A - (C \cup B) \\ f(x) & x \in C \cup B. \end{cases}$$

Then by the Homework 12 problem 4b) (see its solution), h is a bijective function from A to $A - B$.

4. (a) Prove that if A is a countable subset of real numbers, then $|\mathbb{R} - A| = |\mathbb{R}|$.
Hint. Modify the previous hint as follows. Choose $b_n \in (n, n+1) - A$ for all $n \in \mathbb{N}$. (Why does such a b_n exist?) Hence $B = \{b_n : n \in \mathbb{N}\}$ is a countable set of reals disjoint from A. Now choose a bijection $f : A \cup B \to B$. (Explain why this exists using an earlier question.) Finally extend f to a bijection from $\mathbb{R}$ to $\mathbb{R} - A$.
(b) Prove that $|\mathbb{I}| = |\mathbb{R}|$, where $\mathbb{I}$ is the set of irrationals as usual.

Solution: (a). As in the Hint, let us choose $b_n \in (n, n+1) - A$ for all $n \in \mathbb{N}$. Such b_n exists for every n, since if it didn't exist for some value $n = k$, it would mean that A contains the interval $(k, k+1)$; then A would be uncountable since the interval is uncountable – a contradiction. Thus, we have $B = \{b_n : n \in \mathbb{N}\}$, which is a countably infinite set of reals disjoint from A. Now choose a bijection $f : A \cup B \to B$. (Given any two denumerable sets, there exists a bijection between them: suppose $g_1 : \mathbb{N} \to B$ is a bijection; suppose $g_2 : \mathbb{N} \to A \cup B$ is a bijection; then $g_1 \circ g_2^{-1} : A \cup B \to B$ is a bijection). Now define $g(x) : \mathbb{R} \to \mathbb{R} - A$ by:

$$g(x) := \begin{cases} x & \text{if } x \notin A \cup B \\ f(x) & \text{if } x \in A \cup B. \end{cases}$$

By the Lemma from the solution of Problem 2, applied to the identity function from $\mathbb{R} - A \cup B$ to $\mathbb{R} - A \cup B$ and the function f from $A \cup B$ to B, the function g we defined is a bijection from $\mathbb{R} = (\mathbb{R} - (A \cup B)) \cup (A \cup B)$ to $\mathbb{R} - A = (\mathbb{R} - (A \cup B)) \cup B$, and we are done.

(b) Apply the statement of part (a) to $A = \mathbb{Q}$. In particular, note that this statement says that in terms of cardinality, "there are more irrational numbers than rational numbers": we have $|\mathbb{Q}| = |\mathbb{N}| = \aleph_0$, and we just proved that $|\mathbb{I}| = |\mathbb{R}| = c$, and we proved that $\aleph_0 < c$.

5. Let $A = \{0,1\}^{\mathbb{N}}$ be the set of all possible sequences of 0's and 1's. Prove that A is uncountable.

Solution: Suppose P was countable. This means, we would have been able to list all the elements of P as $P = \{p_1, p_2, \dots\}$. By definition of P, each element p_i is a sequence of 0s and 1s. Let us make a new sequence $a = a_1 a_2 a_3 \dots$, defined by $a_k = 0$ if the sequence p_k has a 1 in the kth place, and $a_k = 1$ if the sequence p_k has a 0 in the kth place. This new sequence is also an element of P, and it cannot coincide with any of the sequences p_k, because its kth term is different. We have arrived at the contradiction. Note that this is very similar to the proof of uncountability of the set of real numbers that we discussed in class. This is Cantor's 'diagonal argument'.

6. Let $A = \{0,1\}^{\mathbb{N}}$ be the set of all possible sequences of 0's and 1's, as in the previous problem. Prove that $|A \times A| = |A|$.

Solution: Define $f : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by

$$f(\{x_n\}, \{y_n\})_m = \begin{cases} x_{(m+1)/2} & \text{if } m \text{ is odd} \\ y_{m/2} & \text{if } m \text{ is even.} \end{cases}$$

We claim f is bijective. Let $\{z_n\} \in \{0,1\}^{\mathbb{N}}$. Define $x_n = z_{2n-1}$ and $y_n = z_{2n}$. Then the above definitions easily imply that $\{z_n\} = f(\{x_n\}, \{y_n\})$. Therefore f is onto. Assume $f(\{x_n\}, \{y_n\} = f(\{x'_n\}, \{y'_n\})$. Equating the even coefficients of these two sequences we find that $\{x_n\} = \{x'_n\}$ and equating the odd coefficients we get $\{y_n\} = \{y'_n\}$. Therefore $(\{x_n\}, \{y_n\}) = (\{x'_n\}, \{y'_n\})$, and so f is one-to-one.

7. (a) Prove that the cardinality of the set from the previous problem is, in fact, continuum. **Hint: use binary representation of real numbers**.

Solution: We discussed this in the last class: it is essentially the binary representation of real numbers. Recall how it works. We want to represent every point in the interval $(0,1)$ by a sequence of zeroes and 1s (instead of the decimal representation that uses ten digits). We do it as follows: let $x \in (0,1)$. Divide the interval in half. If x is in the left half, put 0 in the first place after the "decimal point". If x is in the right half, put 1. Then take the sub-interval that contains x and divide it in half. If x is to the left of the middle, put 0 in the second "decimal" place, if x is to the right, put 1, and so on. There will again be a problem, similar to the issue of 0.999... = 1 in the decimal representation: namely, the rational numbers of the form $\frac{a}{2^k}$ where $a, k \in \mathbb{N}$ will have two representations: one finite (or with a "tail of all zeroes"), and one with "a tail of ones". For example,

$$\frac{1}{2} = 0.1 = 0.01111....$$

in the binary representation. Let $B = \{x \in (0,1) : x = \frac{a}{2^k} \text{ for some } a, k \in \mathbb{N}\}$ - the set of "bad points", and let C be the set of sequences of 0s and 1s that are either all zero starting from some place, or all 1 starting from some place (the set of " bad sequences" that correspond to the "bad points"). Then our procedure establishes a bijective correspondence between the sets $(0,1) - B$ and $A - C$. Now note that both B and C are countable (make sure you can prove this!). Then by Problem 3 above, $|(0,1)| = |A|$.

(b) Prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

Solution: By part (a), there exists a bijective function $f : \mathbb{R} \to A$, where A is the set of sequences of 0s and 1s as above. Then the function $g : \mathbb{R} \times \mathbb{R} \to A \times A$ defined by $g(x, y) = (f(x), f(y))$ is also bijective. (Make sure you can prove this!) By Problem 6 above, we know that $A \times A| = |A|$. Then we have:

$$|\mathbb{R} \times \mathbb{R}| = |A \times A| = |A| = |\mathbb{R}|,$$

and we are done!

(c) What is wrong with the following sketch of a proof that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$:
let $\mathbb{R} = \{a_1, \dots, a_n, \dots\}$. *Then* $\mathbb{R} \times \mathbb{R}$ *is the set of pairs* (a_i, b_j) *where* $i, j \in \mathbb{N}$, *and now we can arrange these pairs in a table, and use the algorithm for traversing the table to put them into one long list* $\{c_1, \dots c_n, \dots\}$.
(Note that this proof works fine for $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q} \times \mathbb{Q}$).

Solution: $\mathbb{R}$ is uncountable, so you cannot list its elements!

Definition. Let A be an arbitrary set and let B be a subset of A. Define the function $\chi_B : A \to \{0, 1\}$ by the formula

$$\chi_B(x) := \begin{cases} 0 \text{ if } x \notin B, \\ 1 \text{ if } x \in B. \end{cases}$$

This function is called the *characteristic function* of B.

8. Let $A = \{0, 1\}^{\mathbb{N}}$ be the set of all possible sequences of 0's and 1's. Find a very natural bijective function between A and $\mathcal{P}(\mathbb{N})$.
(**Hint:** construct a function from $\mathcal{P}(\mathbb{N})$ to P, whose definition uses the notion of the characteristic function).

Solution: Consider the function $f : \mathcal{P}(\mathbb{N}) \to P$ defined by $f(A) = \chi_A$, where A is a subset of $\mathbb{N}$ (that is, an element of $\mathcal{P}(N)$, and χ_A is its characteristic function (which can be thought if as a sequence of 0s and 1s). It is easy to check that f is both injective and surjective (we discussed this in the review session).

9. (a) If $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of finite subsets of $\mathbb{N}$, show that $\mathcal{P}_{\text{fin}}(\mathbb{N})$ is countable.

   (b) If $\mathcal{P}_{\text{inf}}(\mathbb{N})$ denotes the set of infinite subsets of $\mathbb{N}$, show that $\mathcal{P}_{\text{inf}}(\mathbb{N})$ is uncountable.

   Hint: Use the previous problem.

Solution: (a) Let A_n denote the set of subsets of $\mathbb{N}$ that are contained in $\{1, \ldots, n\}$. Then $|A_n| = 2^n$ (we proved this by induction in an earlier homework). Now, note that any finite subset of $\mathbb{N}$ is contained in $\{1, \ldots, n\}$ for a large enough number n. Therefore $\mathcal{P}_{\text{fin}}(\mathbb{N}) = \cup_{n=1}^{\infty} A_n$ is a countable union of finite sets and is clearly an infinite set. By Problem 12 from Homework 12, this set is countable.

(b) Assume to the contrary that $B_1 = \mathcal{P}_{\text{inf}}(\mathbb{N})$ is countable. By (a) $B_2 = \mathcal{P}_{\text{fin}}(\mathbb{N})$ is countable. If $B_n = \emptyset$ for all $n \geq 3$ then $\mathcal{P}(\mathbb{N}) = \cup_{n=1}^{\infty} B_n$ is a countable union of countable sets and so by Problem 12 from Homework is also countable. This contradicts the fact that $\mathcal{P}(\mathbb{N})$ is uncountable (see the two previous problems). Therefore the set of infinite subsets of $\mathbb{N}$ must be uncountable.