1. (a) Prove that any number is congruent to the sum of its digits modulo 9.
   Hint: consider a two-digit number $\overline{ab}$ first, and use the definition of congruence.

   (b) Prove that any number is congruent to *alternating sum* of it digits $\mod 11$. The alternating sum is: $a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n$, where a_0 is the last digit (the number of units).

Solution: Let the integer a be written with the digits $a_n, a_{n-1}, \ldots, a_0$: that is,

$$a = \overline{a_n a_{n-1} \ldots a_1 a_0}.$$

Note that a_0 is the number of units, a_1 is he number of 10s, etc., and so a is in fact expressed as:

$$a = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10a_1 + a_0 = \sum_{k=0}^{n} a_k 10^k.$$

(If you are confused by this notation, consider an example: $879 = 8 \cdot 100 + 7 \cdot 10 + 9$; in this example $n = 2$).

In part (a), we need to prove that $a \equiv (a_0 + a_1 + \cdots + a_n) \mod 9$. In Part (b), we need to prove $a \equiv a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n \mod 11$.

Let us start with part (a). By definition of congruence, we need to verify that $a - (a_0 + a_1 + \cdots + a_n)$ is divisible by 9. Using the expression for a, we get:

$$\begin{aligned} &a - (a_0 + \cdots + a_n) = \left(10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10a_1 + a_0\right) - (a_0 + \cdots + a_n) \\ &= \sum_{k=0}^{n} a_k 10^k - \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (10^k - 1) a_k. \end{aligned}$$

It remains to observe that for every $k \geq 0$, $10^k - 1$ is divisible by 9 (make sure you know how to prove it! For example, you can use the same technique we used before: 10 is congruent to $1 \mod 9$, so any power of 10 is congruent to that power of 1, i.e. to 1). Thus, every term of the sum is divisible by 9, and the whole sum is divisible by 9.

Part (b) is prove similarly. We have:

$$a - (a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n) = \sum_{k=0}^{n} a_k 10^k - \sum_{k=0}^{n} a_k (-1)^k = \sum_{k=0}^{n} a_k (10^k - (-1)^k).$$

It remains to note that for every $k \geq 0$, 10^k is congruent to $(-1)^k \mod 11$. Try to prove this fact yourself (and seek help if you cannot!) and also make sure you see how to complete the proof of Part (b) from here.

2. (a) Prove that n is even iff n^3 is even. (*Hint: The contrapositive would be very helpful in one direction.)*

Solution: ($\Rightarrow$) Assume n is even. Then $n \equiv 0 \pmod 2$ and so $n^3 \equiv 0^3 = 0 \pmod 2$. Therefore n^3 is even.

($\Leftarrow$) We prove the contrapositive, that is, n odd$\Rightarrow$ n^3 odd. Assume n is odd. Then $n \equiv 1 \pmod 2$ and so $n^3 \equiv 1^3 = 1 \pmod 2$. That is n^3 is odd.

(b) Prove that $2^{1/3}$ is irrational. *Hint: Use proof by contradiction.*

> **Solution:** Proof by contradiction. Assume to the contrary that $2^{1/3}$ is rational. Then there are integers a and b so that $2^{1/3} = \frac{a}{b}$ and at least one of a and b is odd (by cancelling factors of 2). Cubing both sides and doing a bit of algebra, we get $a^3 = 2b^3$ and so a^3 is even. By (a) above this implies $a = 2k$ is even. Therefore $2^3k^3 = 2b^3$ and so $b^3 = 4k^3$ is also even. By (a) b is also even. This contradicts the fact that at least one of a or b is odd. $\square$.

3. Using induction, prove that the number written as 111...1 (with an even number of 1s) is always divisible by 11.

> **Solution:** We induct on k where $n = 2k$ is the number of 1's. Let a_k be the number with $2k$ 1's. Consider first $k = 1$. Then $a_1 = 11$ which is divisible by 11.
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> Assume now that a_k is divisible by 11 and consider a_{k+1}. By the induction hypothesis $a_k = 11m$ for some natural number m. We have
>
> $$a_{k+1} = 100a_k + 11 = 100(11m) + 11 = 11(100m + 1).$$
>
> Therefore a_{k+1} is divisible by 11. By the Principle of Mathematical Induction, the result holds for all natural numbers k.

4. There are n students in the class. Two volunteers are needed to help deliver a large box of exams to another building. Using induction, prove that there are $n(n-1)/2$ possible choices for the pair of volunteers.

> **Solution:** There are two possible solutions:
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> **Solution 1.** There are n possibilities for choosing the first volunteer – you can pick any one of the n students. For each of these possibilities, there are $n-1$ possible choices of the second volunteer – you can pick anyone except for the person already picked. So it looks like we have $n(n-1)$ possible pairs of volunteers. But now note that we have counted every pair twice: indeed, if we picked Alice first and Bob second, we get Alice and Bob; if we picked Bob first and Alice second, we get Bob and Alice, which are exactly the same pair of volunteers. So we need to divide by 2. This explains the answer.
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> **Solution 2.** By induction on n. Base case: $n = 2$. Then we have to grab both of these people, so we have only one choice, and $2(2-1)/2 = 1$, so it checks out.
>
> Induction step: Suppose we know that if there are n students in class, then there are $n(n-1)/2$ possible choices of a pair of volunteers. We need to prove that if there are $n+1$ students, then there are $n(n+1)/2$ possible ways to choose a pair of them. Let us prove this. Take a student aside, let's say her name is Alice. Then without Alice, there are n people left. There are two possibilities: either Alice volunteers or not. If Alice doesn't volunteer, then we are left wit n people to choose from, and by induction assumption, we have $n(n-1)/2$ ways to choose a pair.
>
> Now, if Alice does volunteer, then we need just one more volunteer from the remaining n people, which gives us n more possibilities. Altogether, we get
>
> $$\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

possibilities, QED.

5. Three friends - Adam, Bob and Cindy - are studying at UBC. One is in math, one is in physics, and one is in chemistry. If Adam is a mathematician, then Cindy is not a physicist. If Bob is not a physicist, then Adam is a mathematician. If Cindy is not a mathematician, then Bob is a chemist.

   Determine which friend is studying which subject.

Solution: Suppose that Cindy is not a mathematician. By the third statement, Bob is a chemist. Since neither Cindy nor Bob is a mathematician, Adam is a mathematician. So, it remains that Cindy is a physicist. However, the first statement is not satisfied because Adam is a mathematician, and Cindy is a physicist. So, Cindy must be a mathematician.

Adam is not a mathematician (because Cindy already is one), so by the contrapositive of the second statement, Bob must be a physicist.

So, Cindy is a mathematician, Bob is a physicist, and Adam is a chemist.