A detailed list of topics to review for the Math 220 final exam, April 2016.

Chapter 1:

(1) The notions of a set, subset, element. Please make sure that you clearly understand the distinction between the usage of the symbols '$\in$' and '$\subseteq$', and also do not confuse things such as $\emptyset$ and $\{\emptyset\}$. In particular, please check that you understand why writing $\{1\} \subseteq \{1, 2, 3\}$ is correct, and $1 \in \{1, 2, 3\}$ is correct, but $1 \subseteq \{1, 2, 3\}$ is incorrect.
(2) The notion of a power set.
(3) Set operations: union, intersection, difference, complement. Venn diagrams.

Chapter 2:

(1) Statements.
(2) Logical operations – conjunction, disjunction, negation.
(3) Open sentences. You should understand the relationship between open sentences and sets: let $P(x)$ be an open sentence, where the domain for the variable x is a set U. Let us think of U as the universal set. Then, we can associate a subset of U with the statement P – it is the set A_P defined as

$$A_P := \{x \in U \mid P(x) \text{is true}\}.$$

Thus, every open sentence *partitions* the domain into two disjoint sets: the set A_P of all elements x such that $P(x)$ is true, and the set of all x such that $P(x)$ is false. With this correspondence between sets and sentences, conjunction of statements corresponds to the intersection of sets, disjunction corresponds to union of sets, and negation corresponds to taking the complement. You should understand this statement.
(4) Logical equivalences and truth tables. You should be able to construct truth tables for compound statements, and use them to check if two compound statements are logically equivalent.
(5) Some important logical equivalences, for example, DeMorgan laws (both for statements and sets). This means, you should be able to correctly negate complex statements involving conjunctions, disjunctions and negations. Distributivity laws.
(6) The implication. Converse and contrapositive. The fact that $P \Rightarrow Q$ is logically equivalent to $\sim P \vee Q$. Logical equivalence of the implication and the contrapositive.
Please note: there are 4 different things:
(i) the implication $P \Rightarrow Q$;

(ii) the converse $Q \Rightarrow P$;
(iii) the contrapositive $\sim Q \Rightarrow \sim P$; and
(iv) the negation of the implication itself, $\sim (P \Rightarrow Q)$.

Of these, only the implication (i) and its contrapositive (iii) are logically equivalent. Otherwise, none of these implies the other. In particular, if $P \Rightarrow Q$ is true, it does not at all mean that the converse should be true (see if you can come up with examples). The negation of the implication $\sim (P \Rightarrow Q)$ is logically equivalent to $\sim (\sim P \vee Q) \equiv P \wedge \sim Q$. In particular, please do not confuse this with the contrapositive or converse, or the contrapositive to the converse (which is a very popular mistake – write down the contrapositive to the converse, and you will see why). You should be able to come up with verbal examples, such as "if I do not buy a lottery ticket, I will not win the lottery". Try making the converse, contrapositive, and the negation of this statement, and look at the differences.

(7) The biconditional.
(8) Quantifiers. Unions and intersections (and complements) of indexed unions of sets. Negation of statements involving quantifiers.

Chapter 3:

(1) Direct proof
(2) Proof by contrapositive
(3) Disproving statements by a counterexample.

You need to know when it is appropriate to use an example to prove or disprove a statement. (If you are using an example to *disprove* a statement, it is called *counterexample*). Look at your Midterm 1 to check if you made a mistake in Problem 2, and make sure you do not make such mistakes again.

Chapter 4:

(1) Basic proofs involving sets (Section 4.4). You need to know the definitions: for example, what does it mean that $A \subseteq B$? Or how to prove that two sets A and B are equal? (you need to prove both inclusions $A \subseteq B$ and $B \subseteq A$).
(2) Basic properties of set operations (Section 4.5); DeMorgan laws for sets.
(3) Cartesian products of sets (section 4.6). For example, you need to feel comfortable with the following definition: let $A_1, \ldots A_n$ be sets. Then $A_1 \times A_2 \times \cdots \times A_n$ is the set of n-tuples of elements

$(a_1, \dots a_n)$ with $a_1 \in A_1, \dots, a_n \in A_n$; in other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \dots a_n) : a_i \in A_i \text{ for } 1 \leq i \leq n\}.$$

(Make sure you understand all the notation in this definition).

(4) Congruences of integers. Operations with congruences.

Chapter 5:

(1) Proof that a statement is false by means of a counterexample. (section 5.1).
(2) Proof by contradiction (Sections 5.2-5.3). In particular, proof by contradiction can include proofs involving rational/irrational numbers.
(3) You need to know how to prove that there are infinitely many prime numbers.

Chapter 6:

(1) The notion of a well-ordered set (you need to know the definition, and be able to come up with examples of sets that are well-ordered, and sets that are not well-ordered). (section 6.1)
(2) Well-ordering of $\mathbb{N}$, and the principle of mathematical induction. You need to know and understand the statements of the well-ordering axiom, and of the principle of mathematical induction and its variations (e.g. strong induction).
(3) You need to be able to prove various statements using induction or strong induction. The types of statements include: formulas for sums; inequalities; statements about divisibility or congruence; statements about sets (such as the cardinality of the set of all subsets of a given set); recursively defined sequences. This is the material in Sections 6.2 and and 6.4, but please also look at your lecture notes and the workshops.

Section 8.1 and Chapter 9:

(1) The definition of a relation (it is a subset of $A \times B$).
(2) The formal definition of a function. Domain and range.
(3) The notion of injective, surjective, bijective function.
(4) Image and inverse image (pre-image) of a set under a function.
(5) Composition of functions.

Chapter 10:

(1) The definition of what it means that $|A| \leq |B|$.
(2) The definition of $|A| = |B|$.
(3) The notion of a denumerable set; countable sets.

(4) The theorems that an infinite set A is denumerable if and only if there exists an injective function from A to $\mathbb{N}$, and if and only of there exists a surjective function from $\mathbb{N}$ to A.
(5) The fact that the union of finitely many denumerable sets is denumerable (with proof).
(6) The fact that the union of countably many denumerable sets is denumerable (with proof).
(7) The fact that direct product of denumerable sets is denumerable (with complete proof, including the proof that $\mathbb{N} \times \mathbb{N}$ is denumerable).
(8) The proof that $\mathbb{Q}$ is denumerable.
(9) The fact that for any set A, $|A| < |\mathcal{P}(A)|$, with complete proof.
(10) The fact that the set $\mathcal{P}(\mathbb{N})$ is uncountable, with proof.
(11) The statement of Schroeder-Bernstein theorem (without proof).
(12) The fact that any interval is uncountable (with proof).
(13) The fact that for any interval (a, b), $|(a, b)| = |\mathbb{R}|$ (need to know how to make a bijection explicitly); the definition of the continuum.

Graph theory:

(1) The definition of a simple graph.
(2) The fact that the number of edges in a simple graph is half the sum of degrees of its vertices. The corollary that the sum of the degrees of the vertices in a graph has to be even (the Handshake Theorem), and the reformulation of this statement: a graph cannot have an odd number of vertices of odd degree.
(3) Connectedness of graphs, the notion of a connected component of a graph.
(4) The notion of an Euler walk in a graph. The fact that a connected graph has an Euler tour if and only if all its vortices have even degree, and has an Euler walk if and only if it has no more than two (that is, 0 or 2) vertices of odd degree.