Notes on indexed collections of sets, and quantifiers.

Please read Section 1.4 about Indexed Collections of Sets. In this handout (as in lecture), we give a precise definition for the union and intersection of an indexed collection of sets.

Suppose we have a collection of sets A_s, indexed by elements $s \in S$ of some set S. For example, if $S = \{1, 2\}$, then we only have two sets A_1 and A_2. If $S = \mathbb{N}$ – the set of natural numbers, then $\{A_n\}_{n\in\mathbb{N}}$ is an infinite collection of sets $A_1, A_2, \dots A_n, \dots$. If $S = \mathbb{R}$ is the set of all real numbers, it means we have a set A_r for every number $r \in \mathbb{R}$. In class, we considered the collection $A_r = [0, r] \times [0, r] \subset \mathbb{R} \times \mathbb{R}$ of squares of size r on the plane – for every positive real number r, we have a square A_r. So in this example, one could say that the indexing set S is the set of all positive real numbers: $S = \{r \in \mathbb{R} \mid r > 0\}$.

How to define the union of an indexed collection of sets? The union has to be the set of elements contained *in at least one of the sets* of the collection.

Definition 1. Let $\{A_s\}_{s\in S}$ be an indexed collection of sets, indexed by the elements of some set S. Then the union of this collection is the set

$$\bigcup_{s\in S} A_s = \{x \mid \exists s \in S, \text{ such that } x \in A_s\}.$$

On the other hand, the intersection of sets is the set of their *common elements* – so it has to be the set of elements that belong to *all* the members of our collection of sets. So we arrive at the definition of the intersection:

Definition 2. The intersection of the collection A_s is the set

$$\bigcap_{s\in S} A_s = \{x \mid \forall s \in S, x \in A_s\}.$$

Exercise: read Section 1.4 again, and see why these definitions agree exactly with the definitions and examples in the book.

Note that these definitions (of course) agree well with DeMorgan laws: the complement of the union should be the intersection of complements. Let us see why this holds for indexed collections as well:

$$\overline{\bigcup_{s\in S} A_s} = \overline{\{x \mid \exists s \in S, \text{ such that } x \in A_s\}}$$

$$= \{x \mid \nexists s \in S, \text{ such that } x \in A_s\} = \{x \mid \forall s \in S, x \notin A_s\} = \bigcap_{s\in S} \overline{A_s}.$$

Exercise: make sure you understand every equality above.