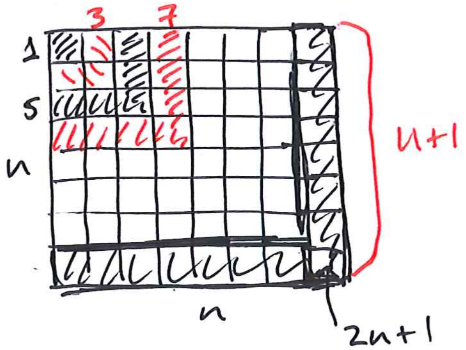
Today: induction

Proposition: The sum of the first n odd numbers is n^2. (from last class).

Will prove by induction.

For now, illustration:



Idea: to get from a square with side n to a square with side $n+1$, we need to add $2n+1$ little squares.

Formal statement: $1+3+5+\dots+2n-1=n^2$ (want to prove it for all $n \in \mathbb{N}$)

Proof: base case: plug in $n=1$ into both sides:

$1 + \dots$

$2 \cdot 1 - 1$, so when $n=1$, the sum has only one term: 1

Right-hand side: 1^2

So we have $1=1$ - checked! ✓

Induction step: induction assumption: the equality holds for n ← some given integer.

Prove it for $n+1$.

Popular mistake: "assume it is true for ~~all n~~ any n"

NO! we need to prove it!

Better: Assume the statement for $n=m$ ← some integer.

Prove it for $n=m+1$

Proof of the induction step:

Recall: $1+3+\cdots+(2n-1)=n^2$ want to prove.

For $n=m$, induction assumption says:

$$1+3+5+\cdots+(2m-1)=m^2$$ ← let $n=m$ in the Proposition.

Want to prove:

Prop. has $2n-1$. I plug in $n=m+1$.

$$1+3+5+\cdots+(2m-1)+\overbrace{(2\cdot(m+1)-1)}=(m+1)^2$$

got this from plugging in $n=m+1$ into the statement of the Proposition.

Rewrite what we want to prove:

$$\underbrace{1+3+\cdots+(2m-1)}_{\underset{m^2}{\shortparallel}}+(2m+1)=(m+1)^2$$

← by induction assumption.

Want to prove: $m^2+(2m+1)=(m+1)^2$ — true by observation.

So we are done.

- By the principle of Mathematical induction, our statement is true for all n.

Worksheet 9: Induction

1. Prove that Bernoulli's inequality: let $x > 0$ be a fixed positive real number. Prove that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

2. Prove that

$$1^2 + 2^2 + \dots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. *All Powers of 2 Are Equal to 1*

   We are going to prove by induction that,

$$\text{For all integers } n \geq 0, \quad 2^n = 1$$

   The claim is verified for $n = 0$; for indeed, $2^0 = 1$.

   Assume the equation is correct for all $n \leq k$, that is

$$2^0 = 1, 2^1 = 1, 2^2 = 1, ..., 2^k = 1.$$

   From these we now derive that also $2^{k+1} = 1$:

$$2^{k+1} = \frac{2^{2k}}{2^{k-1}} = \frac{2^k \times 2^k}{2^{k-1}} = \frac{1 \times 1}{1} = 1$$

   Induction is complete. *Where is the error?*

4. Using induction, prove that the number written as 111...1 (with an even number of 1s) is always divisible by 11.

(1) x is fixed.

Induction on n.

base case: $n=1$: $(1+x)^1 \geq 1+1\cdot x$

$1+x = 1+x$ - true.

Induction step: Assume true for $n=m$.
Need to prove for $n=m+1$.

This means, assume: $(1+x)^m \geq 1+mx$.

Want to prove: $(1+x)^{m+1} \geq 1+(m+1)x$

Scratchwork

$$
\begin{array}{ccc}
(1+x)^{m+1} & & 1+(m+1)x \\
\| & & \| \\
(1+x)^m\cdot(1+x) & & 1+mx+x \\
\vee\!| & & \\
(1+mx)(1+x) & & \\
\| & & \\
1+mx+x+mx^2 & &
\end{array}
$$

by induction assumption

Get: we want to prove: $1+mx+x+mx^2 \geq 1+mx+x$

which is true b/c $mx^2 \geq 0$.

Write the proof in better order:

by induction assumption, $(1+x)^m \geq 1+mx$.

Multiply both sides by $1+x$, which is positive, so we get:

$$(1+x)^m(1+x) \geq (1+mx)(1+x)$$
$$= 1+mx+x+mx^2 = 1+(m+1)x+mx^2$$
$$\geq 1+(m+1)x, \text{ as required.}$$

(2) Base case: $n = 1$

$$1 = \frac{1(1+1)(2\cdot 1+1)}{6} = 1 \quad \text{- true.}$$

Induction step: Assume the statement is true for $n = m$:

$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}$$

Want to prove:

plug in $n = m+1$ into the question

$$\sum_{k=1}^{m+1} k^2 = \boxed{\frac{(m+1)(m+1+1)(2(m+1)+1)}{6}}$$

Proof of induction step:

$$\sum_{k=1}^{m+1} k^2 = \underbrace{1+4+9+\dots+m^2}_{\frac{m(m+1)(2m+1)}{6}} + (m+1)^2$$

$\frac{m(m+1)(2m+1)}{6}$ by induction assumption.

Want to prove:

$$\frac{m\cancel{(m+1)}(2m+1)}{\cancel{6}} + (m+1)^2 = \boxed{\frac{\cancel{(m+1)}(m+2)(2m+3)}{\cancel{6}}}$$

$\Leftrightarrow$

cancel 6 and $m+1$. → since $m+1 \neq 0$

$$m(2m+1) + 6(m+1) = (m+2)(2m+3)$$

$$\Leftrightarrow \quad 2m^2 + m + 6m + 6 = 2m^2 + 4m + 3m + 6$$

True!

Then we are done by the principle of Math Induction.

<u>Strong induction</u> - variation where the induction assumption assumes not just that the statement is true for $n = m$, but actually also true for <u>all</u> $n \le m$

(i.e. assume it for $n = 1, 2, \ldots, m$.)

induction step is not $P(m) \Rightarrow P(m+1)$

but:

$$\underbrace{(P(1) \wedge P(2) \wedge \cdots \wedge P(m))}_{\text{strong induction assumption.}} \Rightarrow P(m+1)$$

Fine to use!

Very useful for things like <u>Fibonacci numbers</u>: (see homework)

<u>defined</u>: $F_1 = 1$, $F_2 = 1$ $\quad F_n = F_{n-1} + F_{n-2} \quad n \ge 3$

$1, 1, 2, 3, 5, 8, 13, \ldots$

When proving things by induction, you need the assumption not just for $n-1$, but for $n-2$ <u>as well</u>.

Want <u>strong induction</u>

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$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. *All Powers of 2 Are Equal to 1*

   We are going to prove by induction that,
$$\text{For all integers } n \geq 0, \quad 2^n = 1$$
   The claim is verified for $n = 0$; for indeed, $2^0 = 1$. starting from $n=0$ NOT $n=-1$!

   Assume the equation is correct for all $n \leq k$, that is
$$2^0 = 1, 2^1 = 1, 2^2 = 1, \ldots, 2^k = 1.$$
   From these we now derive that also $2^{k+1} = 1$:
$$2^{k+1} = \frac{2^{2k}}{2^{k-1}} = \frac{2^k \times 2^k}{2^{k-1}} = \frac{1 \times 1}{1} = 1$$
   Induction is complete. *Where is the error?*

   uses 2^{k-1} so for all k, the statement about 2^{k-1} needs to be true. base case $n=0$, also need it for $n=-1$ then.

   $2^{-1} \neq 1$

   so actually the step from $n=0$ to $n=1$ doesn't work.

4. Using induction, prove that the number written as 111...1 (with an even number of 1s) is always divisible by 11.