Worksheet 9: Induction

1. Prove that Bernoulli's inequality: let $x > 0$ be a fixed positive real number. Prove that $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

2. Prove that
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. *All Powers of 2 Are Equal to 1*

   We are going to prove by induction that,
$$\text{For all integers } n \geq 0, \quad 2^n = 1$$
   The claim is verified for $n = 0$; for indeed, $2^0 = 1$.

   Assume the equation is correct for all $n \leq k$, that is
$$2^0 = 1, 2^1 = 1, 2^2 = 1, ..., 2^k = 1.$$
   From these we now derive that also $2^{k+1} = 1$:
$$2^{k+1} = \frac{2^{2k}}{2^{k-1}} = \frac{2^k \times 2^k}{2^{k-1}} = \frac{1 \times 1}{1} = 1$$
   Induction is complete. *Where is the error?*

4. Using induction, prove that the number written as 111...1 (with an even number of 1s) is always divisible by 11.