Homework 5: Linear transformations; matrices. Part 2.
Due Monday February 27.

1. Consider the linear space V of degree n polynomials over a field $F \subset \mathbb{C}$ (that is, the space of all functions $f : F \to F$ of the form $f(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, \ldots, a_n \in F$).

   (a) Find the dimension of the space V.

   (b) Let $D : V \to V$ be the linear map $D(f) = f'$ (the derivative). Find the kernel and image of D.

2. Let $C(\mathbb{R})$ be the space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Let $D(f) = f'' + f$. Show that D is a linear operator on the space $C(R)$, and describe its kernel. Is its kernel finite-dimensional? Make a guess at its dimension (you do not have to include a rigorous proof, but explain your guess.)

3. Let V be an arbitrary vector space over a field F, and let $P : V \to V$ be a linear operator with the property that $P^2 = P$ (here by P^2 we mean P composed with itself). Such linear operators are called *projectors.*

   (a) Prove that $V = \mathrm{Ker}(P) \oplus \mathrm{Im}(P)$.

   (b) Make an example of such a linear operator on $\mathbb{R}^3$.

4. Problem 5.1 from Jänisch

5. Problem 5.2 from Jänisch

6. Problem 5.3 from Jänisch

7. Problem 7.1 from Jänisch. In addition, find a basis for the space $\mathrm{Ker}(A)$ for the matrix of this system of equations.

8. Problem 7.2 from Jänisch. In addition, find a basis for the space $\mathrm{Ker}(A)$ for the matrix of this system of equations.